

CAREERS 360

PRACTICE **Series**

UK Board Class 12

Mathematics

**Previous Year Questions
with Detailed Solution**

UK Board Class 12 Maths Question with Solution- 2024

i) If $\sin^{-1} x = y$, then -

- (i) $0 \leq y \leq \pi$
- (ii) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- (iii) $0 < y < \pi$
- (iv) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Solution:

The given equation is $\sin^{-1} x = y$, which means that y is the inverse sine (also called arcsine) of x . The function $\sin^{-1} x$ returns values in a specific range of the principal value of the sine function. The range of the arcsine function is:

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

This means that the correct option is:

(ii) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Thus, the value of y lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive.

ii) Let A be a non-singular square matrix of order 3×3 . Then $|AdjA|$ is equal to-

- (i) $|A|$
- (ii) $|A|^2$
- (iii) $|A|^3$
- (iv) $3|A|$

Solution:

Given that A is a non-singular square matrix of order 3×3 , we are asked to find the value of $|\text{adj}(A)|$, where $\text{adj}(A)$ denotes the adjugate (or adjoint) of A .

The property of the determinant of the adjugate matrix is given by:

$$|\text{adj}(A)| = |A|^{n-1}$$

where n is the order of the square matrix. For a 3×3 matrix, $n = 3$, so:

$$|\text{adj}(A)| = |A|^{3-1} = |A|^2$$

Thus, the correct answer is:

(ii) $|A|^2$.

iii) Differentiation of $\sin(\cos(x^2))$ with respect to ' x ' will be -

- (i) $\cos(\cos(x^2))$
- (ii) $\cos(\sin 2x)$
- (iii) $2x \cos(\sin(x^2))$
- (iv) $-2x \cos(\cos x^2) \sin x^2$

Solution:

To solve this, we'll use the chain rule:

1. Let $f(x) = \sin(\cos(x^2))$.
2. The outer function is $\sin(u)$, and the derivative of $\sin(u)$ is $\cos(u)$, where $u = \cos(x^2)$.
3. The inner function is $\cos(x^2)$, and the derivative of $\cos(x^2)$ is $-\sin(x^2) \cdot 2x$ (applying the chain rule to $\cos(x^2)$).

Now, applying the chain rule step by step:

$$\frac{d}{dx} \sin(\cos(x^2)) = \cos(\cos(x^2)) \cdot \frac{d}{dx}(\cos(x^2))$$

$$\frac{d}{dx}(\cos(x^2)) = -\sin(x^2) \cdot 2x$$

Thus, the derivative becomes:

$$\frac{d}{dx} \sin(\cos(x^2)) = \cos(\cos(x^2)) \cdot (-2x \sin(x^2))$$

$$= -2x \cos(\cos(x^2)) \sin(x^2)$$

Therefore, the correct answer is:

(iv) $-2x \cos(\cos(x^2)) \sin(x^2)$.

iv) If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is -

- (i) $x^4 + \frac{1}{x^3} - \frac{129}{8}$
- (ii) $x^3 + \frac{1}{x^4} + \frac{129}{8}$
- (iii) $x^4 - \frac{1}{x^3} + \frac{129}{8}$
- (iv) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Solution:

Step 1: Integrate $\frac{d}{dx} f(x)$

$$f(x) = \int (4x^3 - \frac{3}{x^4}) dx$$

Integrating $4x^3$:

$$\int 4x^3 dx = \frac{4x^4}{4} = x^4$$

Integrating $-\frac{3}{x^4}$:

$$\int -\frac{3}{x^4} dx = \frac{3}{3x^3} = -\frac{1}{x^3}$$

So, the general form of $f(x)$ is:

$$f(x) = x^4 - \frac{1}{x^3} + C$$

Step 2: Use the condition $f(2) = 0$

Substitute $x = 2$ and $f(2) = 0$:

$$0 = 2^4 - \frac{1}{2^3} + C$$

$$0 = \downarrow j - \frac{1}{8} + C$$

$$0 = 16 - \frac{1}{8} + C$$

$$0 = 16 - \frac{1}{8} + C \implies C = -16 + \frac{1}{8} = -\frac{128}{8} + \frac{1}{8} = -\frac{127}{8}$$

Step 3: Write the final function

Thus, the function $f(x)$ is:

$$f(x) = x^4 - \frac{1}{x^3} - \frac{127}{8}$$

Therefore, the correct answer is:

(iii) $x^4 - \frac{1}{x^3} + \frac{129}{8}$.

v) $\int \sqrt{1+x^2} dx$ is equal to -

(i) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x + \sqrt{1+x^2}) + c$

(ii) $\frac{2}{3}(1+x^2)^{3/2} + c$

(iii) $\frac{2}{3} \times (1+x^2)^{3/2} + c$

(iv) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2 \log(x + \sqrt{1+x^2}) + c$

Solution:

The integral $\int \sqrt{1+x^2} dx$ can be solved using the substitution method, leading to the result:

$$\int \sqrt{1+x^2} dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x + \sqrt{1+x^2}) + c$$

Thus, the correct option is (i).

vi) The number of arbitrary constants in general solution of a differential equation of fourth order are

-

(i) 4

(ii) 3

(iii) 2

(iv) 0

Solution:

The number of arbitrary constants in the general solution of a differential equation is equal to the order of the differential equation. Since the given differential equation is of fourth order, the number of arbitrary constants is:

(i) 4.

vii) Equation of a line parallel to x -axis and passing through the origin is -

(i) $\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$

(ii) $\frac{x}{0} = \frac{y}{1} = \frac{z}{1}$

(iii) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

(iv) $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

Solution:

The equation of a line parallel to the x -axis and passing through the origin would have no change in the y - and z -coordinates, while x can vary. Therefore, the correct form of the equation is:

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

Thus, the correct answer is:

(iii) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

viii) If $P(B) = 0.5$ and $P(A \cap B) = 0.3$, then the value of conditional probability $P(A/B)$ will be -

- (i) 0.6
- (ii) 0.2
- (iii) 0.15
- (iv) 0.8

Solution:

The conditional probability $P(A/B)$ is given by the formula:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Substituting the given values:

$$P(A/B) = \frac{0.3}{0.5} = 0.6$$

Therefore, the correct answer is:

- (i) 0.6 .

Direction : In next two parts of Question No.-1, there are two statements labelled as Assertion (A) and Reason (R). From the following options (i), (ii), (iii) and (iv), select their correct answer.

(i) Both A and R are correct and R is the correct explanation of A .

(ii) Both A and R are correct but R is not the correct explanation of A .

(iii) A is correct but R is incorrect.

(iv) Both A and R are incorrect.

(ix) Assertion (A) : The function $f : \mathbf{Z} \rightarrow \mathbf{Z}$, given by $f(x) = 2x$ is one-one. Reason (R) : Function f is not onto.

Solution:

- Assertion (A): The function $f : \mathbf{Z} \rightarrow \mathbf{Z}$, given by $f(x) = 2x$, is one-one.

This assertion is correct because the function $f(x) = 2x$ assigns a unique output for every unique input x . Therefore, the function is injective (one-one).

- Reason (R): The function f is not onto.

This reason is also correct because $f(x) = 2x$ maps integers to only even integers. Hence, not every integer in the codomain \mathbf{Z} has a preimage in the domain (since odd integers are not covered), so the function is not onto.

Thus, both Assertion (A) and Reason (R) are correct, but R does not explain why $f(x)$ is one-one.

Therefore, the correct answer is:

(ii) Both A and R are correct, but R is not the correct explanation of A .

(x) Assertion (A) : Points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - 3\hat{k})$ are collinear.

Solution:

- Assertion (A): Points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$, and $C(7\hat{i} - 3\hat{k})$ are collinear. To check for collinearity, we need to verify if the vectors \overrightarrow{AB} and \overrightarrow{AC} are scalar multiples of each other. Let's calculate the vectors:

$$\overrightarrow{AB} = B - A = (1 + 2)\hat{i} + (2 - 3)\hat{j} + (3 - 5)\hat{k} = 3\hat{i} - \hat{j} - 2\hat{k}$$

$$\overrightarrow{AC} = C - A = (7 + 2)\hat{i} + (0 - 3)\hat{j} + (-3 - 5)\hat{k} = 9\hat{i} - 3\hat{j} - 8\hat{k}$$

Since \overrightarrow{AC} is not a scalar multiple of \overrightarrow{AB} , the points are not collinear.

Thus, the Assertion (A) is incorrect.

The Reason (R) is not provided in this case, so we only conclude based on the assertion.

Therefore, the correct answer is:

(iv) Both A and R are incorrect.

2) If $y = A \sin x + B \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

Solution:

Step 1: Differentiate y with respect to x

First, find the first derivative of y :

$$\frac{dy}{dx} = A \cos x - B \sin x$$

Step 2: Differentiate again to find $\frac{d^2y}{dx^2}$

Now, differentiate $\frac{dy}{dx}$ with respect to x :

$$\frac{d^2y}{dx^2} = -A \sin x - B \cos x$$

Step 3: Add $\frac{d^2y}{dx^2}$ and y

Now, let's add $\frac{d^2y}{dx^2}$ and y :

$$\frac{d^2y}{dx^2} + y = (-A \sin x - B \cos x) + (A \sin x + B \cos x)$$

Simplifying the right-hand side:

$$\frac{d^2y}{dx^2} + y = -A \sin x - B \cos x + A \sin x + B \cos x = 0$$

Thus, we have proved that:

$$\frac{d^2y}{dx^2} + y = 0$$

3) Find $\frac{dy}{dx}$, if $x = 2at^2$, $y = at^4$

Solution:

Step 1: Differentiate x with respect to t

$$\frac{dx}{dt} = 2a \cdot 2t = 4at$$

Step 2: Differentiate y with respect to t

$$\frac{dy}{dt} = 4at^3$$

Step 3: Find $\frac{dy}{dx}$

By the chain rule, we know that:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

Step 4: Express t^2 in terms of x

From the equation $x = 2at^2$, we can solve for t^2 :

$$t^2 = \frac{x}{2a}$$

Step 5: Substitute back into $\frac{dy}{dx}$

$$\frac{dy}{dx} \downarrow t^2 = \frac{x}{2a}$$

4) Evaluate $\int \frac{1}{x^2+36} dx$

Solution:

We are tasked with evaluating the integral:

$$\int \frac{1}{x^2+36} dx$$

This integral is of the standard form:

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Here, $a^2 = 36$, so $a = 6$. Applying the formula, we get:

$$\int \frac{1}{x^2+36} dx = \frac{1}{6} \tan^{-1} \left(\frac{x}{6} \right) + C$$

Thus, the evaluated integral is:

$$\frac{1}{6} \tan^{-1} \left(\frac{x}{6} \right) + C$$

5) Find the general solution of differential equation $\frac{dy}{dx} = e^{x-y}$.

Solution:

Step 1: Rearrange the equation

We rewrite the equation as:

$$\frac{dy}{dx} = e^x e^{-y}$$

This can be rearranged to separate the variables x and y :

$$e^y dy = e^x dx$$

Step 2: Integrate both sides

Now, integrate both sides:

$$\int e^y dy = \int e^x dx$$

The integrals are:

$$\int e^y dy = e^y + C_1 \quad \text{and} \quad \int e^x dx = e^x + C_2$$

Thus, we get:

$$e^y = e^x + C$$

where $C = C_2 - C_1$ is the constant of integration.

Step 3: Solve for y

To express y explicitly, take the natural logarithm of both sides:

$$y = \ln(e^x + C)$$

Final solution:

The general solution of the differential equation is:

$$y = \ln(e^x + C)$$

6) Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution:

Step 1: Write the vector in component form

The given vector is:

$$\mathbf{v} = x(\hat{i} + \hat{j} + \hat{k}) = x\hat{i} + x\hat{j} + x\hat{k}$$

Step 2: Find the magnitude of the vector

The magnitude of the vector \mathbf{v} is:

$$|\mathbf{v}| = \sqrt{(x)^2 + (x)^2 + (x)^2} = \sqrt{3x^2} = |x|\sqrt{3}$$

Step 3: Set the magnitude equal to 1

Since the vector is a unit vector, we set the magnitude equal to 1 :

$$|x|\sqrt{3} = 1$$

Solving for x :

$$|x| = \frac{1}{\sqrt{3}}$$

$$\text{Thus, } x = \frac{1}{\sqrt{3}} \text{ or } x = -\frac{1}{\sqrt{3}}.$$

7) Find the direction ratios of a line joining the points $(2, 3, -4)$ and $(1, 3, -2)$.

Solution:

The direction ratios of a line are simply the differences in the coordinates of the two points on the line. Given the points $P(2, 3, -4)$ and $Q(1, 3, -2)$, we can find the direction ratios as the differences between the corresponding coordinates of these points.

For the points $P(x_1, y_1, z_1) = (2, 3, -4)$ and $Q(x_2, y_2, z_2) = (1, 3, -2)$, the direction ratios (l, m, n) are:

$$l = x_2 - x_1 = 1 - 2 = -1$$

$$m = y_2 - y_1 = 3 - 3 = 0$$

$$n = z_2 - z_1 = -2 - (-4) = -2 + 4 = 2$$

Thus, the direction ratios of the line joining the points $(2, 3, -4)$ and $(1, 3, -2)$ are:

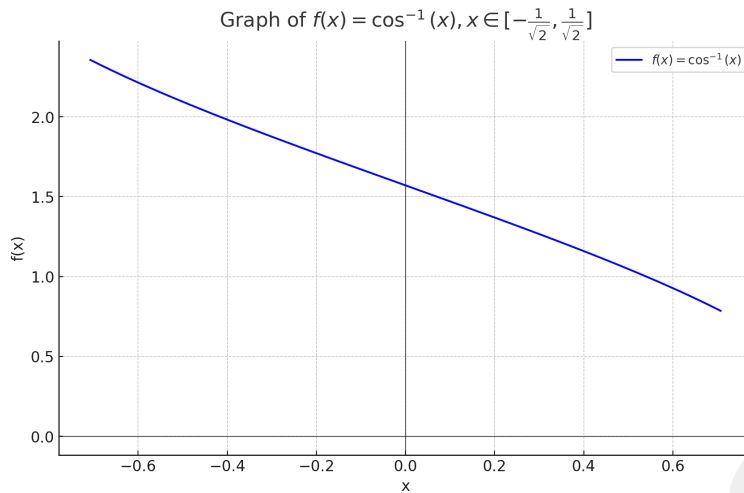
$$(-1, 0, 2)$$

8) Draw the graph of function $f(x) = \cos^{-1} x$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. Also write range of function $f(x)$.

Solution:

The range of the function $f(x) = \cos^{-1}(x)$, where $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, is $\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right), \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Here is the range, and I attempted to create a graph for you, but it didn't generate properly. You can visualize this graph by plotting $\cos^{-1}(x)$ between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, where the range would correspond to values between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.



10) Find $\int \frac{dx}{(x+2)(x+3)}$

Solution:

Step 1: Use Partial Fraction Decomposition

We express the integrand as a sum of partial fractions:

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

Multiply both sides by $(x+2)(x+3)$ to eliminate the denominators:

$$1 = A(x+3) + B(x+2)$$

Now, expand the right-hand side:

$$1 = A(x) + 3A + B(x) + 2B$$

Group like terms:

$$1 = (A+B)x + (3A+2B)$$

This gives us two equations:

$$1. A + B = 0$$

$$2. 3A + 2B = 1$$

Step 2: Solve for A and B

From the first equation, $A = -B$.

Substitute this into the second equation:

$$3(-B) + 2B = 1$$

$$-3B + 2B = 1$$

$$-B = 1 \Rightarrow B = -1$$

Now, substitute $B = -1$ into $A = -B$:

$$A = 1$$

Step 3: Rewrite the Integral

Using the values of A and B , we rewrite the integrand:

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

Step 4: Integrate

Now, integrate each term:

$$\int \frac{1}{x+2} dx - \int \frac{1}{x+3} dx$$

The integrals are:

$$\ln|x+2| - \ln|x+3| + C$$

Step 5: Combine the Logarithms

Using properties of logarithms:

$$\ln \frac{x+2}{x+3} + C$$

Thus, the final result is:

$$\int \frac{dx}{(x+2)(x+3)} = \ln \frac{x+2}{x+3} + C$$

$$11) \frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ / and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

Solution:

The given system represents two parametric forms of straight lines in 3D space. Let's break them down:

Line 1:

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$$

This is the symmetric form of a line. We can express the parametric equations for this line as:

$$x = 2t, \quad y = 2t, \quad z = t$$

where t is a parameter.

Line 2:

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

Similarly, this is the symmetric form of another line. The parametric equations for this line are:

$$x = 4s + 5, \quad y = s + 2, \quad z = 8s + 3$$

where s is a parameter.

12) A random variable x has the following probability distribution. Determine the value of $P(x < 3)$.

x	0	1	2	3	4
P(x)	0	k	2k	4k	3k

Solution:

Step 1: Use the fact that the sum of probabilities must equal 1

Since $P(x)$ is a valid probability distribution, the sum of all probabilities must equal 1. Thus, we have:

$$P(0) + P(1) + P(2) + P(3) + P(4) = 1$$

Substitute the values:

$$0 + k + 2k + 4k + 3k = 1$$

Simplify:

$$10k = 1$$

Solve for k :

$$k = \frac{1}{10}$$

Step 2: Calculate $P(x < 3)$

The probability that $x < 3$ means $P(x < 3) = P(0) + P(1) + P(2)$. From the given distribution, we know:

$$P(0) = 0, \quad P(1) = k, \quad P(2) = 2k$$

Thus:

$$P(x < 3) = 0 + k + 2k = 3k$$

Substitute $k = \frac{1}{10}$:

$$P(x < 3) = 3 \times \frac{1}{10} = \frac{3}{10}$$

Final Answer:

$$P(x < 3) = \frac{3}{10}$$

13) Show that $f : \mathbf{N} \rightarrow \mathbf{N}$, given by $f(x) = \begin{cases} x + 1, & \text{if } x \text{ is odd} \\ x - 1, & \text{if } x \text{ is even} \end{cases}$ is both one-one and onto.

Solution:

1. Proving that the function is one-one (injective):

To prove injectivity, we need to show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

- Case 1: If both x_1 and x_2 are odd, then:

$$f(x_1) = x_1 + 1 \quad \text{and} \quad f(x_2) = x_2 + 1$$

If $f(x_1) = f(x_2)$, we have:

$$x_1 + 1 = x_2 + 1$$

This implies $x_1 = x_2$, so the function is injective in this case.

- Case 2: If both x_1 and x_2 are even, then:

$$f(x_1) = x_1 - 1 \quad \text{and} \quad f(x_2) = x_2 - 1$$

If $f(x_1) = f(x_2)$, we have:

$$x_1 - 1 = x_2 - 1$$

This implies $x_1 = x_2$, so the function is injective in this case as well.

- Case 3: If one of x_1 and x_2 is odd and the other is even, then:

If x_1 is odd and x_2 is even, we have:

$$f(x_1) = x_1 + 1 \quad \text{and} \quad f(x_2) = x_2 - 1$$

Clearly, $x_1 + 1 \neq x_2 - 1$, so $f(x_1) \neq f(x_2)$ in this case.

Since in all cases, $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is injective.

2. Proving that the function is onto (surjective):

To prove surjectivity, we need to show that for every $y \in \mathbf{N}$, there exists an $x \in \mathbf{N}$ such that

$$f(x) = y.$$

- Case 1: If y is even, then $y = 2k$ for some integer k . To satisfy $f(x) = y$, let $x = y + 1$ (which is odd). Then:

$$f(x) = x + 1 = y$$

So, for every even y , there exists an odd x such that $f(x) = y$.

- Case 2: If y is odd, then $y = 2k + 1$ for some integer k . To satisfy $f(x) = y$, let $x = y - 1$ (which is even). Then:

$$f(x) = x - 1 = y$$

So, for every odd y , there exists an even x such that $f(x) = y$.

Since every $y \in \mathbf{N}$ has a corresponding $x \in \mathbf{N}$ such that $f(x) = y$, the function is onto.

Conclusion:

The function $f(x)$ is both one-one (injective) and onto (surjective), and hence it is a bijective function.

$$14) \text{ If } F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ show that } F(x) \cdot F(y) = F(x + y).$$

Solution:

Step 1: Compute $F(x) \cdot F(y)$

First, write down $F(x)$ and $F(y)$:

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, perform matrix multiplication $F(x) \cdot F(y)$:

$$F(x) \cdot F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing the matrix multiplication (row-by-column):

- First row:

$$\begin{aligned} \cos x \cdot \cos y + (-\sin x) \cdot \sin y &= \cos x \cos y - \sin x \sin y \\ \cos x \cdot (-\sin y) + (-\sin x) \cdot \cos y &= -\cos x \sin y - \sin x \cos y = -(\sin(x + y)) \\ \cos x \cdot 0 + (-\sin x) \cdot 0 &= 0 \end{aligned}$$

- Second row:

$$\begin{aligned} \sin x \cdot \cos y + \cos x \cdot \sin y &= \sin x \cos y + \cos x \sin y = \sin(x + y) \\ \sin x \cdot (-\sin y) + \cos x \cdot \cos y &= -\sin x \sin y + \cos x \cos y = \cos(x + y) \end{aligned}$$

$$\sin x \cdot 0 + \cos x \cdot 0 = 0$$

- Third row:

$$0 \cdot \cos y + 0 \cdot \sin y = 0$$

$$0 \cdot (-\sin y) + 0 \cdot \cos y = 0$$

$$0 \cdot 0 + 1 \cdot 1 = 1$$

Thus, the product $F(x) \cdot F(y)$ is:

$$F(x) \cdot F(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: Compare with $F(x+y)$

The matrix $F(x+y)$ is given by:

$$F(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we can see that:

$$F(x) \cdot F(y) = F(x+y)$$

15) For which value of k , the following given function is continuous at $x = \pi/2$:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \pi/2 \\ 3, & \text{if } x = \pi/2 \end{cases}$$

Solution:

To determine the value of k for which the function $f(x)$ is continuous at $x = \pi/2$, we need to ensure that the limit of $f(x)$ as $x \rightarrow \pi/2$ equals the function's value at $x = \pi/2$.

1. Function Definition:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \pi/2 \\ 3, & \text{if } x = \pi/2 \end{cases}$$

2. Continuity Condition at $x = \pi/2$:

For continuity at $x = \pi/2$, we require:

$$\lim_{x \rightarrow \pi/2} f(x) = f\left(\frac{\pi}{2}\right) = 3$$

3. Evaluate the Limit $\lim_{x \rightarrow \pi/2} f(x)$:

$$\lim_{x \rightarrow \pi/2} \frac{k \cos x}{\pi - 2x}$$

Substitute $x = \pi/2$:

Direct substitution results in a 0/0 indeterminate form because:

$$- \cos\left(\frac{\pi}{2}\right) = 0, \text{ and}$$

$$- \pi - 2\left(\frac{\pi}{2}\right) = 0.$$

So, we apply L'Hopital's rule.

4. Apply L'Hopital's Rule:

Differentiate the numerator and the denominator with respect to x :

- Numerator: The derivative of $k \cos x$ is $-k \sin x$.

- Denominator: The derivative of $\pi - 2x$ is -2 .

Thus:

$$\lim_{x \rightarrow \pi/2} \frac{-k \sin x}{-2} = \lim_{x \rightarrow \pi/2} \frac{k \sin x}{2}$$

5. Evaluate the Limit:

$$\lim_{x \rightarrow \pi/2} \frac{k \sin x}{2} = \frac{k \cdot \sin(\frac{\pi}{2})}{2} = \frac{k \cdot 1}{2} = \frac{k}{2}$$

6. Set the Limit Equal to the Function Value:

For the function to be continuous at $x = \pi/2$, we require:

$$\frac{k}{2} = 3$$

16) For which value of k , the following given function is continuous at $x = \pi/2$:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \pi/2 \\ 3, & \text{if } x = \pi/2 \end{cases}$$

Solution:

Step 1: Evaluate $f(\frac{\pi}{2})$

From the given piecewise function, we have:

$$f\left(\frac{\pi}{2}\right) = 3$$

Step 2: Compute the limit as $x \rightarrow \frac{\pi}{2}$

We need to compute the limit of $f(x)$ as $x \rightarrow \frac{\pi}{2}$ from the expression for $f(x)$ when $x \neq \frac{\pi}{2}$:

$$f(x) = \frac{k \cos x}{\pi - 2x}$$

Taking the limit as $x \rightarrow \frac{\pi}{2}$:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

Substitute $x = \frac{\pi}{2}$ into $\cos x$ and $\pi - 2x$:

$$- \cos\left(\frac{\pi}{2}\right) = 0$$

$$- \pi - 2 \times \frac{\pi}{2} = 0$$

Thus, we encounter the indeterminate form $\frac{0}{0}$, so we apply L'Hôpital's Rule.

Step 3: Apply L'Hôpital's Rule

Differentiate the numerator and denominator with respect to x :

- Derivative of the numerator $k \cos x$ is $-k \sin x$.

- Derivative of the denominator $\pi - 2x$ is -2 .

Now, applying L'Hôpital's Rule:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{-k \sin x}{-2} = \frac{k \sin(\frac{\pi}{2})}{2} = \frac{k \cdot 1}{2} = \frac{k}{2}$$

Step 4: Set the limit equal to the function value at $x = \frac{\pi}{2}$

For continuity, the limit must equal the value of the function at $x = \frac{\pi}{2}$:

$$\frac{k}{2} = 3$$

Solve for k :

$$k = 6$$

17 Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution:

To find a vector perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, we can take the cross product of these two vectors.

1. Compute $\vec{a} + \vec{b}$:

$$\vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k}) = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

2. Compute $\vec{a} - \vec{b}$:

$$\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 0\hat{i} - \hat{j} - 2\hat{k}$$

3. Now, compute the cross product $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$:

$$\vec{v} = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times (0\hat{i} - \hat{j} - 2\hat{k})$$

Using the determinant method for cross product:

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

Expanding the determinant:

$$\vec{v} = \hat{i} \begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

Simplifying:

$$\vec{v} = \hat{i}((3)(-2) - (4)(-1)) - \hat{j}((2)(-2) - (4)(0)) + \hat{k}((2)(-1) - (3)(0))$$

$$\vec{v} = \hat{i}(-6 + 4) - \hat{j}(-4) + \hat{k}(-2)$$

$$\vec{v} = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

4. Now, find the magnitude of this vector:

$$|\vec{v}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

5. Finally, the unit vector is:

$$\hat{v} = \frac{-2\hat{i}+4\hat{j}-2\hat{k}}{2\sqrt{6}} = \frac{-\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$$

Thus, the unit vector perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is:

$$\frac{-\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$$

18. Find the cartesian equation of the line passing through the point $(1, 2, -4)$ and perpendicular to each of the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Solution:

To find the line perpendicular to both given lines, we must find the direction ratios of the desired line by taking the cross product of the direction ratios of the two given lines.

1. The direction ratios of the first line are $(3, -16, 7)$.
2. The direction ratios of the second line are $(3, 8, -5)$.

Now, compute the cross product of these two direction ratios:

$$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

Expanding the determinant:

$$\vec{d} = \hat{i} \begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 7 \\ 3 & -5 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix}$$

Simplifying:

$$\begin{aligned} \vec{d} &= \hat{i}((-16)(-5) - (7)(8)) - \hat{j}((3)(-5) - (7)(3)) + \hat{k}((3)(8) - (3)(-16)) \\ \vec{d} &= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48) \\ \vec{d} &= \hat{i}(24) - \hat{j}(-36) + \hat{k}(72) \end{aligned}$$

$$\vec{d} = 24\hat{i} + 36\hat{j} + 72\hat{k}$$

The direction ratios of the required line are $(24, 36, 72)$, which can be simplified to $(2, 3, 6)$.

The parametric form of the line passing through the point $(1, 2, -4)$ and having direction ratios $(2, 3, 6)$ is:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Thus, the cartesian equation of the line is:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

19. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is 70. Find the cost of each item per kg by matrix method.

Solution:

Let the cost per kg of onion, wheat, and rice be x , y , and z , respectively.

We have the following system of equations:

$$4x + 3y + 2z = 60$$

$$2x + 4y + 6z = 90$$

$$6x + 2y + 3z = 70$$

This system can be written in matrix form as:

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Let A be the coefficient matrix, X the variable matrix, and B the constant matrix:

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

20)

Solve, given system of linear equations, using matrix method:

$$5x + 2y = 4$$

$$7x + 3y = 5$$

Solution:

Step 1: Matrix Representation

The given system of linear equations is:

$$5x + 2y = 4$$

$$7x + 3y = 5$$

In matrix form, this can be written as:

$$A \cdot X = B$$

where:

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Step 2: Find the Inverse of Matrix A

The inverse of matrix A , denoted as A^{-1} , is calculated using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

- Determinant of A :

$$\det(A) = 5 \cdot 3 - 7 \cdot 2 = 15 - 14 = 1$$

21) A wire of length 28 cm is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Solution:

Step 1: Variables and expressions

Let:

- x be the length of the wire used to make the square.
- $28 - x$ be the length of the wire used to make the circle.

1. Square:

- The perimeter of the square is x , so the side length of the square is $\frac{x}{4}$.
- The area of the square is:

$$A_{\text{square}} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

2. Circle:

- The circumference of the circle is $28 - x$, so the radius of the circle is:

$$r = \frac{28-x}{2\pi}$$

- The area of the circle is:

$$A_{\text{circle}} = \pi r^2 = \pi \left(\frac{28-x}{2\pi}\right)^2 = \frac{(28-x)^2}{4\pi}$$

Step 2: Total area function

The total area of the square and the circle is:

$$A(x) = A_{\text{square}} + A_{\text{circle}} = \frac{x^2}{16} + \frac{(28-x)^2}{4\pi}$$

Step 3: Minimize the total area

To minimize the total area, we differentiate $A(x)$ with respect to x and set the derivative equal to zero.

First, differentiate $A(x)$:

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(28-x)(-1)}{4\pi} = \frac{x}{8} - \frac{28-x}{2\pi}$$

Set the derivative equal to zero:

$$\frac{x}{8} - \frac{28-x}{2\pi} = 0$$

Multiply through by 8π to eliminate the denominators:

$$\pi x - 4(28 - x) = 0$$

Simplify:

$$\pi x - 112 + 4x = 0$$

$$(\pi + 4)x = 112$$

Solve for x :

$$x = \frac{112}{\pi+4}$$

Step 4: Lengths of the pieces

The length of the wire used for the square is $x = \frac{112}{\pi+4}$.

The length of the wire used for the circle is:

$$28 - x = 28 - \frac{112}{\pi+4}$$

Thus, the two pieces are:

- Square piece length: $\frac{112}{\pi+4}$
- Circle piece length: $28 - \frac{112}{\pi+4}$

23) Find the particular solution satisfying the given conditions for the following differential equation :

$$(1 + x^2) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0 \text{ when } x = 1$$

Solution:

Step 1: Recognize the form of the equation

This is a first-order linear differential equation. We can rewrite it in the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

First, divide both sides of the equation by $1 + x^2$:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$$

Now, it is in the form $\frac{dy}{dx} + P(x)y = Q(x)$, where:

$$P(x) = \frac{2x}{1+x^2}, \quad Q(x) = \frac{1}{(1+x^2)^2}$$

Step 2: Find the integrating factor

The integrating factor $\mu(x)$ is given by:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx}$$

Since $\int \frac{2x}{1+x^2}dx$ is a standard integral, we know:

$$\int \frac{2x}{1+x^2}dx = \ln(1 + x^2)$$

Thus, the integrating factor is:

$$\mu(x) = e^{\ln(1+x^2)} = 1 + x^2$$

Step 3: Multiply the equation by the integrating factor

Multiply both sides of the equation by $\mu(x) = 1 + x^2$:

$$(1 + x^2) \frac{dy}{dx} + \frac{2x}{1+x^2} (1 + x^2)y = \frac{1+x^2}{(1+x^2)^2}$$

Simplifying, this becomes:

$$\frac{d}{dx} ((1 + x^2)y) = \frac{1}{1+x^2}$$

Step 4: Integrate both sides

Now, integrate both sides with respect to x :

$$\int \frac{d}{dx} ((1 + x^2)y) dx = \int \frac{1}{1+x^2} dx$$

The left-hand side simplifies to:

$$(1 + x^2)y$$

The right-hand side is a standard integral:

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x)$$

Thus, we have:

$$(1 + x^2)y = \tan^{-1}(x) + C$$

Step 5: Solve for y

Solving for y , we get:

$$y = \frac{\tan^{-1}(x) + C}{1 + x^2}$$

Step 6: Use the initial condition to find C

We are given that $y(1) = 0$. Substituting $x = 1$ and $y = 0$ into the equation:

$$0 = \frac{\tan^{-1}(1) + C}{1 + 1^2}$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$, this gives:

$$0 = \frac{\frac{\pi}{4} + C}{2}$$

Multiplying by 2 :

$$0 = \frac{\pi}{4} + C$$

Thus:

$$C = -\frac{\pi}{4}$$

Step 7: Final solution

Substitute $C = -\frac{\pi}{4}$ into the expression for y :

$$y = \frac{\tan^{-1}(x) - \frac{\pi}{4}}{1 + x^2}$$

Thus, the particular solution is:

21) A wire of length 28 cm is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

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- The circumference of the circle is $28 - x$, so the radius of the circle is:

$$r = \frac{28 - x}{2\pi}$$

- The area of the circle is:

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To minimize the total area, we differentiate $A(x)$ with respect to x and set the derivative equal to zero.

First, differentiate $A(x)$:

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(28-x)(-1)}{4\pi} = \frac{x}{8} - \frac{28-x}{2\pi}$$

Set the derivative equal to zero:

$$\frac{x}{8} - \frac{28-x}{2\pi} = 0$$

Multiply through by 8π to eliminate the denominators:

$$\pi x - 4(28 - x) = 0$$

Simplify:

$$\pi x - 112 + 4x = 0$$

$$(\pi + 4)x = 112$$

Solve for x :

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Step 4: Lengths of the pieces

The length of the wire used for the square is $x = \frac{112}{\pi+4}$.

The length of the wire used for the circle is:

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Thus, the two pieces are:

- Square piece length: $\frac{112}{\pi+4}$

- Circle piece length: $28 - \frac{112}{\pi+4}$

23) Find the particular solution satisfying the given conditions for the following differential equation :

$$(1 + x^2) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0 \text{ when } x = 1$$

Solution:

Step 1: Recognize the form of the equation

This is a first-order linear differential equation. We can rewrite it in the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

First, divide both sides of the equation by $1 + x^2$:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$$

Now, it is in the form $\frac{dy}{dx} + P(x)y = Q(x)$, where:

$$P(x) = \frac{2x}{1+x^2}, \quad Q(x) = \frac{1}{(1+x^2)^2}$$

Step 2: Find the integrating factor

The integrating factor $\mu(x)$ is given by:

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Since $\int \frac{2x}{1+x^2} dx$ is a standard integral, we know:

$$\int \frac{2x}{1+x^2} dx = \ln(1+x^2)$$

Thus, the integrating factor is:

$$\mu(x) = e^{\ln(1+x^2)} = 1+x^2$$

Step 3: Multiply the equation by the integrating factor

Multiply both sides of the equation by $\mu(x) = 1+x^2$:

$$(1+x^2) \frac{dy}{dx} + \frac{2x}{1+x^2} (1+x^2)y = \frac{1+x^2}{(1+x^2)^2}$$

Simplifying, this becomes:

$$\frac{d}{dx} ((1+x^2)y) = \frac{1}{1+x^2}$$

Step 4: Integrate both sides

Now, integrate both sides with respect to x :

$$\int \frac{d}{dx} ((1+x^2)y) dx = \int \frac{1}{1+x^2} dx$$

The left-hand side simplifies to:

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$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x)$$

Thus, we have:

$$(1+x^2)y = \tan^{-1}(x) + C$$

Step 5: Solve for y

Solving for y , we get:

$$y = \frac{\tan^{-1}(x)+C}{1+x^2}$$

Step 6: Use the initial condition to find C

We are given that $y(1) = 0$. Substituting $x = 1$ and $y = 0$ into the equation:

$$0 = \frac{\tan^{-1}(1)+C}{1+1^2}$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$, this gives:

$$0 = \frac{\frac{\pi}{4}+C}{2}$$

Multiplying by 2:

$$0 = \frac{\pi}{4} + C$$

Thus:

$$C = -\frac{\pi}{4}$$

Step 7: Final solution

Substitute $C = -\frac{\pi}{4}$ into the expression for y :

$$y = \frac{\tan^{-1}(x) - \frac{\pi}{4}}{1+x^2}$$

Thus, the particular solution is:

$$y = \frac{\tan^{-1}(x) - \frac{\pi}{4}}{1+x^2}$$

24) There is a factory which manufactures LED bulbs. In this factory, three different type of machines A , B and C are installed for manufacturing LED bulbs. Machines A , B and C manufactures 25%, 35% and 40% of the total production respectively. Out of these no machine is hundred percent efficient and of their outputs 5, 4 and 2 percent are respectively defective bulbs. The above events are defined as follows:

L_1 : The LED bulb is manufactured by machine A .

L_2 : The LED bulb is manufactured by machine B

L_3 : The LED bulb is manufactured by machine C .

Clearly, L_1 , L_2 and L_3 events are mutually exclusive and exhaustive. If event ' E ' is defined as the event of 'the bulb is defective', then based on the above information, answer the following questions:

(i) A bulb is drawn at random from the total production, then what is the probability that it is manufactured by the machine A ?

1

(ii) What is the probability of manufacturing a defected bulb by machine C ?

(iii) A bulb is drawn at random from the total production and is found to be defective. What is the probability that it is manufactured by the machine B ?

Solution:

Definitions of Events:

- L_1 : The bulb is manufactured by machine A , with probability $P(L_1) = 0.25$ (since 25% of the production is done by machine A).

- L_2 : The bulb is manufactured by machine B , with probability $P(L_2) = 0.35$ (since 35% of the production is done by machine B).

- L_3 : The bulb is manufactured by machine C , with probability $P(L_3) = 0.40$ (since 40% of the production is done by machine C).

The probabilities of producing defective bulbs are:

- The probability of a defective bulb from machine A is 5%, i.e., $P(E | L_1) = 0.05$.

- The probability of a defective bulb from machine B is 4%, i.e., $P(E | L_2) = 0.04$.

- The probability of a defective bulb from machine C is 2%, i.e., $P(E | L_3) = 0.02$.

Let E be the event that a randomly chosen bulb is defective.

(i) What is the probability that the bulb is manufactured by machine A ?

This is simply the probability that a randomly chosen bulb is manufactured by machine A :

$$P(L_1) = 0.25$$

(ii) What is the probability of manufacturing a defective bulb by machine C ?

This is the conditional probability that a defective bulb comes from machine C , i.e., $P(E | L_3)$:

$$P(E | L_3) = 0.02$$

(iii) What is the probability that a defective bulb was manufactured by machine B ?

We are asked to find $P(L_2 | E)$, the probability that the defective bulb came from machine B . We will use Bayes' Theorem for this:

$$P(L_2 | E) = \frac{P(E|L_2) \cdot P(L_2)}{P(E)}$$

Where $P(E)$ is the total probability of drawing a defective bulb, which can be calculated using the Total Probability Theorem:

$$P(E) = P(E | L_1) \cdot P(L_1) + P(E | L_2) \cdot P(L_2) + P(E | L_3) \cdot P(L_3)$$

Substituting the values:

$$P(E) = (0.05 \cdot 0.25) + (0.04 \cdot 0.35) + (0.02 \cdot 0.40)$$

$$P(E) = 0.0125 + 0.014 + 0.008 = 0.0345$$

Now, applying Bayes' Theorem:

$$P(L_2 | E) = \frac{0.04 \cdot 0.35}{0.0345} = \frac{0.014}{0.0345} \approx 0.4058$$

UK Board Class 12 Maths Question with Solution- 2023

1. (i) Principal value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is-

(1) $\frac{3\pi}{4}$

(ii) $\frac{\pi}{4}$

(iii) $-\frac{\pi}{4}$

(iv) $\frac{\pi}{3}$

Solution:

Principal value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$:

- The principal value of $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is the angle whose cosine is $\frac{1}{\sqrt{2}}$. The angle that satisfies this condition is $\frac{\pi}{4}$ radians.

- Answer: $\frac{\pi}{4}$

(ii) Matrices A and B will be inverse of each other only if-

(i) $AB = BA$

(ii) $AB = BA = 0$

(iii) $AB = 0, BA = 1$

(iv) $AB = BA = 1$

Solution:

Matrices A and B will be inverse of each other only if:

- Two matrices A and B are inverses of each other if and only if $AB = BA = I$, where I is the identity matrix (not 1).

- Answer: $AB = BA = 1$ is incorrect, but the condition $AB = BA = I$ holds.

(iii) Differentiation of $\sin(x^2)$ with respect to ' x ' is-

(i) $\cos(x^2)$

(ii) $2x \sin(x^2)$

(iii) $2x \cos(x^2)$

(iv) $\cos(2x)$

Solution:

Using the chain rule, the derivative of $\sin(x^2)$ with respect to x is $\frac{d}{dx}[\sin(x^2)] = 2x \cos(x^2)$.

Answer: $2x \cos(x^2)$

(iv) If $f(a + b - x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to-

(i) 0

(ii) $\frac{a+b}{2} \int_0^p f(b+x) dx$

(iii) $\frac{b-a}{2} \int_a^b f(x) dx$

(iv) $\frac{a+b}{2} \int_a^b f(x) dx$

Solution:

If $f(a + b - x) = f(x)$, this implies symmetry in the function. As a result, the integral $\int_a^b x f(x) dx$ becomes zero.

Answer: 0

(v) The order of the differential equation $2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0$ is:

(i) 2

(ii) 1

(iii) 0

(iv) 3

Solution:

The order of a differential equation is the highest derivative present. In this case, the highest derivative is $\frac{d^2y}{dx^2}$, which is a second derivative.

Answer: 2

(vi) Direction cosines of the vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ are-

(i) $(\frac{1}{6}, \frac{1}{6}, \frac{-2}{6})$

(ii) $(1, 1, -2)$

(iii) $(\sqrt{6}, \sqrt{6}, -2\sqrt{6})$

(iv) $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}})$

Solution:

The direction cosines of a vector are given by the components of the vector divided by the magnitude of the vector.

The magnitude of \vec{a} is:

$$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

So, the direction cosines are:

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$$

Answer: $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$

(vii) The Cartesian equation of the plane $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ is-

(i) $x + y - z = 0$

(ii) $x + y - z = 2$

(iii) $x + y - z = 1$

(iv) $x + y + z + 2 = 0$

Solution:

The equation of a plane in vector form is given as $\vec{r} \cdot \vec{n} = d$, where \vec{n} is the normal vector and d is the constant on the right-hand side.

In this case, $\vec{n} = \hat{i} + \hat{j} - \hat{k}$ and $d = 2$.

The Cartesian form of the equation is obtained by writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The dot product gives:

$$x + y - z = 2$$

Answer: $x + y - z = 2$

(viii) If A and B are independent events, where probabilities $P(A) = 0.3$ and $P(B) = 0.6$, then value of $P(A \cap B)$ will be-

(i) 1

(ii) 0.18

(iii) 0.9

(iv) 0.01

Solution:

For independent events, the probability of their intersection is given by:

$$P(A \cap B) = P(A) \times P(B)$$

So,

$$P(A \cap B) = 0.3 \times 0.6 = 0.18$$

Answer: 0.18

2. Find the value of $\cos(\sec^{-1} x + \operatorname{cosec}^{-1} x)$, $|x| \geq 1$.

Solution:

Let:

$$\theta = \sec^{-1} x \quad \text{and} \quad \phi = \csc^{-1} x$$

This means:

$$\sec(\theta) = x \quad \text{and} \quad \csc(\phi) = x$$

From these, we have:

$$\cos(\theta) = \frac{1}{x} \quad \text{and} \quad \sin(\phi) = \frac{1}{x}$$

Now, using the trigonometric identity for the sum of angles:

$$\cos(\theta + \phi) = \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi)$$

Since $\sec(\theta) = x$ and $\csc(\phi) = x$, it follows that:

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - \frac{1}{x^2}} = \frac{\sqrt{x^2 - 1}}{x} \quad \text{and} \quad \cos(\phi) = \sqrt{1 - \sin^2(\phi)} = \frac{\sqrt{x^2 - 1}}{x}$$

Now, substituting into the sum formula:

$$\cos(\theta + \phi) = \frac{1}{x} \cdot \frac{\sqrt{x^2 - 1}}{x} - \frac{\sqrt{x^2 - 1}}{x} \cdot \frac{1}{x}$$

This simplifies to:

$$\cos(\theta + \phi) = 0$$

Answer: 0

3. If a matrix has 14 elements, what are the possible orders it can have?

Solution:

The order of a matrix is given by $m \times n$ (rows \times columns), and the total number of elements is $m \times n = 14$.

We need to find pairs of integers m and n such that $m \times n = 14$. The possible pairs are:

$$(1, 14), (14, 1), (2, 7), (7, 2)$$

Answer: The possible orders are 1×14 , 14×1 , 2×7 , and 7×2 .

4. Find dy/dx , if $y + \sin y = \cos x$.

Solution:

To differentiate implicitly, take the derivative of both sides with respect to x :

$$\frac{d}{dx}(y + \sin y) = \frac{d}{dx}(\cos x)$$

Using the chain rule:

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

Factor out $\frac{dy}{dx}$ on the left-hand side:

$$\frac{dy}{dx}(1 + \cos y) = -\sin x$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-\sin x}{1 + \cos y}$$

Answer: $\frac{-\sin x}{1 + \cos y}$

5. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when $x = 15$.

Solution:

Marginal revenue is the derivative of the total revenue function $R(x)$ with respect to x . Differentiate $R(x)$:

$$\frac{dR}{dx} = 6x + 36$$

Now, substitute $x = 15$ to find the marginal revenue:

$$\frac{dR}{dx} = 6(15) + 36 = 90 + 36 = 126$$

Answer: The marginal revenue when $x = 15$ is 126 rupees.

6. Integrate the function $\frac{\tan^{-1}x}{1+x^2}$ with respect to ' x '.

Solution:

We can use substitution to solve this.

Let:

$$u = \tan^{-1}x$$

$$\text{Then, } \frac{du}{dx} = \frac{1}{1+x^2}, \text{ or equivalently, } du = \frac{dx}{1+x^2}.$$

Now the integral becomes:

$$\int u du$$

This is a standard integral:

$$\int u du = \frac{u^2}{2} + C$$

Substitute $u = \tan^{-1}x$ back into the equation:

$$\frac{(\tan^{-1}x)^2}{2} + C$$

$$\text{Answer: } \frac{(\tan^{-1}x)^2}{2} + C$$

7. Find the integral :

\$

$$\int \left(4e^{3x} + 2\right) dx.$$

\$

Solution:

$$\int (4e^{3x} + 2) dx$$

We can split this into two integrals:

$$\int 4e^{3x} dx + \int 2 dx$$

For the first integral, we use the substitution $u = 3x$, so $du = 3dx$, which gives:

$$\int 4e^{3x} dx = \frac{4}{3}e^{3x}$$

For the second integral:

$$\int 2dx = 2x$$

Thus, the total integral is:

$$\frac{4}{3}e^{3x} + 2x + C$$

Answer: $\frac{4}{3}e^{3x} + 2x + C$

8. If $\bar{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\bar{b} = 3\hat{i} + \hat{j} + 3\hat{k}$, then calculate $\bar{a} \cdot \bar{b}$.

Solution:

The dot product $\bar{a} \cdot \bar{b}$ is given by:

$$\bar{a} \cdot \bar{b} = (2\hat{i}) \cdot (3\hat{i}) + (-3\hat{j}) \cdot (\hat{j}) + (\hat{k}) \cdot (3\hat{k})$$

Now compute each term:

$$2 \times 3 = 6, \quad (-3) \times 1 = -3, \quad 1 \times 3 = 3$$

So:

$$\bar{a} \cdot \bar{b} = 6 - 3 + 3 = 6$$

Answer: 6

9. Find the equation of a line parallel to x -axis and passing through the origin.

Solution:

A line parallel to the x -axis has the form $y = c$, where c is the y -coordinate of any point on the line. Since the line passes through the origin $(0, 0)$, we have:

$$y = 0$$

Answer: $y = 0$

10. In a set of rational numbers Q , a binary operation '*' is defined as follows:

\$

$$a * b = a + a b; \quad a, b \in Q$$

\$

Show that '*' is neither commutative nor associative.

Solution:

1. Check if '*' is commutative:

A binary operation is commutative if:

$$a * b = b * a$$

Using the definition of * :

$$a * b = a + ab \quad \text{and} \quad b * a = b + ba$$

Since $a * b = a + ab$ and $b * a = b + ab$, they are not equal unless $a = b$.

Thus, $*$ is not commutative.

2. Check if ' $*$ ' is associative:

A binary operation is associative if:

$$a * (b * c) = (a * b) * c$$

First, compute $a * (b * c)$:

$$b * c = b + bc$$

$$a * (b * c) = a + a(b + bc) = a + ab + abc$$

Now compute $(a * b) * c$:

$$a * b = a + ab$$

$$(a * b) * c = (a + ab) + (a + ab)c = a + ab + ac + abc$$

Since $a * (b * c) = a + ab + abc$ and $(a * b) * c = a + ab + ac + abc$, the two expressions are not equal.

Thus, $*$ is not associative.

11. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x$ is increasing.

Solution:

The first derivative of $f(x)$ is:

$$f'(x) = \frac{d}{dx}(2x^3 - 3x^2 - 36x)$$

Using the power rule for differentiation:

$$f'(x) = 6x^2 - 6x - 36$$

To find the critical points, set $f'(x) = 0$:

$$6x^2 - 6x - 36 = 0$$

Divide through by 6:

$$x^2 - x - 6 = 0$$

Factor the quadratic:

$$(x - 3)(x + 2) = 0$$

The solutions are:

$$x = 3 \quad \text{and} \quad x = -2$$

Now, we perform a sign analysis on $f'(x)$ to determine where the function is increasing (i.e., where $f'(x) > 0$).

- For $x < -2$, choose $x = -3$:

$$f'(-3) = 6(-3)^2 - 6(-3) - 36 = 54 + 18 - 36 = 36 > 0$$

So, $f'(x) > 0$ for $x < -2$.

- For $-2 < x < 3$, choose $x = 0$:

$$f'(0) = 6(0)^2 - 6(0) - 36 = -36 < 0$$

So, $f'(x) < 0$ for $-2 < x < 3$.

- For $x > 3$, choose $x = 4$:

$$f'(4) = 6(4)^2 - 6(4) - 36 = 96 - 24 - 36 = 36 > 0$$

So, $f'(x) > 0$ for $x > 3$.

Conclusion:

The function $f(x) = 2x^3 - 3x^2 - 36x$ is increasing in the intervals:

$$(-\infty, -2) \cup (3, \infty)$$

12. Evaluate $\int_0^{x/2} \cos^2 x dx$

Solution:

To integrate $\cos^2(x)$, we use the identity:

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Thus, the integral becomes:

$$\int_0^{x/2} \cos^2(x) dx = \int_0^{x/2} \frac{1 + \cos(2x)}{2} dx$$

We can split this into two integrals:

$$\frac{1}{2} \int_0^{x/2} 1 dx + \frac{1}{2} \int_0^{x/2} \cos(2x) dx$$

For the first integral:

$$\frac{1}{2} \int_0^{x/2} 1 dx = \frac{1}{2} [x]_0^{x/2} = \frac{1}{2} \times \frac{x}{2} = \frac{x}{4}$$

For the second integral, use substitution. Let $u = 2x$, so $du = 2dx$, and adjust the limits accordingly:

$$\int_0^{x/2} \cos(2x) dx = \frac{1}{2} \int_0^x \cos(u) du = \frac{1}{2} \sin(u) \Big|_0^x = \frac{1}{2} \sin(x)$$

Thus, the full integral is:

$$\frac{x}{4} + \frac{1}{4} \sin(x)$$

13. Find the angle between the two planes $3x - 6y + 2z = 10$ and $2x + 2y - 2z = 15$.

Solution:

The angle between two planes can be found by using the normal vectors of the planes. The direction ratios of the normal vectors to the planes are the coefficients of x , y , and z in the plane equations.

For the first plane $3x - 6y + 2z = 10$, the normal vector is:

$$\vec{n}_1 = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

For the second plane $2x + 2y - 2z = 15$, the normal vector is:

$$\vec{n}_2 = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

The angle θ between the planes is the same as the angle between their normal vectors, and it can be found using the dot product formula:

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

First, compute the dot product $\vec{n}_1 \cdot \vec{n}_2$:

$$\vec{n}_1 \cdot \vec{n}_2 = (3)(2) + (-6)(2) + (2)(-2) = 6 - 12 - 4 = -10$$

Next, find the magnitudes of \vec{n}_1 and \vec{n}_2 :

$$|\vec{n}_1| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$$|\vec{n}_2| = \sqrt{2^2 + 2^2 + (-2)^2} = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$$

Now, use the dot product formula:

$$\cos \theta = \frac{-10}{7 \times 2\sqrt{3}} = \frac{-10}{14\sqrt{3}} = \frac{-5}{7\sqrt{3}}$$

Thus, the angle θ is:

$$\theta = \cos^{-1} \left(\frac{-5}{7\sqrt{3}} \right)$$

$$\text{Answer: } \theta = \cos^{-1} \left(\frac{-5}{7\sqrt{3}} \right).$$

14. In a box containing 100 bulbs, 10 are defective. What is the probability that out of a sample of 5 bulbs, none is defective?

Solution:

The problem is a typical example of a hypergeometric probability. The total number of bulbs is 100, and 10 are defective. We are selecting 5 bulbs, and we want none of them to be defective.

Let's define the following:

- Total bulbs: $N = 100$
- Defective bulbs: $k = 10$
- Sample size: $n = 5$
- We want the number of defective bulbs in the sample to be 0.

The probability of selecting no defective bulbs is given by the hypergeometric probability formula:

$$P(\text{none defective}) = \frac{\binom{90}{5}}{\binom{100}{5}}$$

where $\binom{90}{5}$ represents the number of ways to choose 5 non-defective bulbs out of 90, and $\binom{100}{5}$ represents the number of ways to choose 5 bulbs out of 100.

Now, calculate:

$$\binom{100}{5} = \frac{100 \times 99 \times 98 \times 97 \times 96}{5 \times 4 \times 3 \times 2 \times 1} = 75287520$$

$$\binom{90}{5} = \frac{90 \times 89 \times 88 \times 87 \times 86}{5 \times 4 \times 3 \times 2 \times 1} = 43735320$$

Thus, the probability is:

$$P(\text{none defective}) = \frac{43735320}{75287520} \approx 0.581$$

15. Consider the function $f; \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse function of f .

Solution:

Step 1: Check if $f(x)$ is invertible.

A function is invertible if it is one-to-one (injective) and onto (surjective).

- Injectivity: To check if $f(x)$ is injective, assume $f(x_1) = f(x_2)$, and show that $x_1 = x_2$:

$$4x_1 + 3 = 4x_2 + 3$$

Subtract 3 from both sides:

$$4x_1 = 4x_2$$

Divide by 4:

$$x_1 = x_2$$

Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is injective.

- Surjectivity: Since $f(x) = 4x + 3$ is a linear function, it maps all real numbers onto all real numbers, so it is surjective.

Thus, $f(x)$ is invertible.

Step 2: Find the inverse function.

To find the inverse function $f^{-1}(x)$, we solve for x in terms of y :

$$y = 4x + 3$$

Subtract 3 from both sides:

$$y - 3 = 4x$$

Divide by 4:

$$x = \frac{y-3}{4}$$

Thus, the inverse function is:

$$f^{-1}(x) = \frac{x-3}{4}$$

16) For what value of x , $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$?

Solution:

Step 1: Multiply the second matrix by the column vector.

First, we perform the multiplication of the matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix}$$

The result is obtained by taking the dot product of each row with the column vector:

- First row: $1 \times 0 + 2 \times 2 + 0 \times x = 4$

- Second row: $2 \times 0 + 0 \times 2 + 1 \times x = x$

- Third row: $1 \times 0 + 0 \times 2 + 2 \times x = 2x$

So the result of this multiplication is the column vector:

$$\begin{bmatrix} 4 \\ x \\ 2x \end{bmatrix}$$

Step 2: Multiply the row vector by the result from Step 1.

Now, we need to multiply:

$$[1 \quad 2 \quad 1] \begin{bmatrix} 4 \\ x \\ 2x \end{bmatrix}$$

This is done by taking the dot product of the row vector and the column vector:

$$1 \times 4 + 2 \times x + 1 \times 2x = 4 + 2x + 2x = 4 + 4x$$

Step 3: Set the result equal to 0 .

We are given that the result of the multiplication is equal to 0 :

$$4 + 4x = 0$$

Step 4: Solve for x .

Now, solve the equation for x :

$$4 + 4x = 0$$

$$4x = -4$$

$$x = -1$$

17) If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$.

Solution:

Step 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

First, differentiate x and y with respect to t .

$$\frac{dx}{dt} = a \left(\frac{d}{dt}(\cos t) + t \frac{d}{dt}(\sin t) + \sin t \right)$$

$$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$\frac{dx}{dt} = at \cos t$$

Next, for y :

$$\frac{dy}{dt} = a \left(\frac{d}{dt}(\sin t) - t \frac{d}{dt}(\cos t) - \cos t \right)$$

$$\frac{dy}{dt} = a(\cos t + t \sin t - \cos t)$$

$$\frac{dy}{dt} = at \sin t$$

Step 2: Find $\frac{dy}{dx}$

Now, we compute $\frac{dy}{dx}$ using the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t} = \tan t$$

Step 3: Find $\frac{d^2y}{dx^2}$

To find $\frac{d^2y}{dx^2}$, we need to differentiate $\frac{dy}{dx} = \tan t$ with respect to t and then divide by $\frac{dx}{dt}$.

First, differentiate $\tan t$ with respect to t :

$$\frac{d}{dt}(\tan t) = \sec^2 t$$

Now, divide by $\frac{dx}{dt} = at \cos t$:

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{at \cos t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{at \cos^3 t}$$

18) Evaluate $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$

Solution:

Step 1: Substitution Attempt

We begin by noting that both $\sec x$ and $\tan x$ can be expressed in terms of $\sin x$ and $\cos x$:

$$\sec x = \frac{1}{\cos x}, \quad \tan x = \frac{\sin x}{\cos x}$$

Thus, the integrand becomes:

$$\frac{x \tan x}{\sec x + \tan x} = \frac{x \frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} = \frac{x \sin x}{1 + \sin x}$$

Therefore, the integral simplifies to:

$$I = \int_0^\pi \frac{x \sin x}{1 + \sin x} dx$$

Step 2: Symmetry of the Integral

Next, let's check whether symmetry can help simplify the integral. The limits of integration are from 0 to π , which suggests that we may be able to use properties of the sine function, specifically the fact that $\sin(\pi - x) = \sin x$.

To explore this, we substitute $x = \pi - t$. Under this substitution:

$$dx = -dt, \quad \text{when } x = 0, t = \pi, \quad \text{and when } x = \pi, t = 0$$

So, the integral becomes:

$$I = \int_\pi^0 \frac{(\pi-t) \sin(\pi-t)}{1 + \sin(\pi-t)} (-dt)$$

Since $\sin(\pi - t) = \sin t$, this simplifies to:

$$I = \int_0^\pi \frac{(\pi-t) \sin t}{1 + \sin t} dt$$

Thus, we now have two expressions for the same integral:

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

and

$$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x} dx$$

Step 3: Add the Two Integrals

Let's add these two expressions:

$$2I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx + \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x} dx$$

This simplifies to:

$$2I = \int_0^{\pi} \frac{(x + (\pi - x)) \sin x}{1 + \sin x} dx$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x} dx$$

Thus, the integral becomes:

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

Step 4: Solve the Remaining Integral

Now, let's solve the integral:

$$\int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

We can rewrite this as:

$$\begin{aligned} \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx &= \int_0^{\pi} \left(1 - \frac{1}{1 + \sin x} \right) dx \\ &= \int_0^{\pi} 1 dx - \int_0^{\pi} \frac{1}{1 + \sin x} dx \end{aligned}$$

The first integral is straightforward:

$$\int_0^{\pi} 1 dx = \pi$$

Now we need to evaluate the second integral:

$$\int_0^{\pi} \frac{1}{1 + \sin x} dx$$

This integral is a standard result and equals $\frac{\pi}{2}$.

Thus:

$$\int_0^{\pi} \frac{\sin x}{1 + \sin x} dx = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Step 5: Final Calculation

Substitute this result back into the expression for $2I$:

$$2I = \pi \times \frac{\pi}{2}$$

$$2I = \frac{\pi^2}{2}$$

Therefore:

$$I = \frac{\pi^2}{4}$$

19. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ then find the value of $2\vec{a} - \vec{b}$ and $|\vec{a} \times \vec{b}|$.

Solution:

Step 1: Compute $2\vec{a} - \vec{b}$

We are given:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$$

First, calculate $2\vec{a}$:

$$2\vec{a} = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

Now compute $2\vec{a} - \vec{b}$:

$$\begin{aligned} 2\vec{a} - \vec{b} &= (2\hat{i} + 2\hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) \\ &= (2\hat{i} - 2\hat{i}) + (2\hat{j} - (-\hat{j})) + (2\hat{k} - 3\hat{k}) \\ &= 0\hat{i} + 3\hat{j} - \hat{k} \\ 2\vec{a} - \vec{b} &= 3\hat{j} - \hat{k} \end{aligned}$$

Step 2: Compute $|\vec{a} \times \vec{b}|$

The cross product $\vec{a} \times \vec{b}$ is given by the determinant of the following matrix:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$$

Now, expand the determinant:

$$\vec{a} \times \vec{b} = \hat{i} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

Compute each of the 2×2 determinants:

$$\vec{a} \times \vec{b} = \hat{i}(1 \times 3 - 1 \times (-1)) - \hat{j}(1 \times 3 - 1 \times 2) + \hat{k}(1 \times (-1) - 1 \times 2)$$

$$\vec{a} \times \vec{b} = \hat{i}(3 + 1) - \hat{j}(3 - 2) + \hat{k}(-1 - 2)$$

$$\vec{a} \times \vec{b} = 4\hat{i} - \hat{j} - 3\hat{k}$$

Thus, $\vec{a} \times \vec{b} = 4\hat{i} - \hat{j} - 3\hat{k}$.

Now, compute the magnitude $|\vec{a} \times \vec{b}|$:

$$|\vec{a} \times \vec{b}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$$

Final Answer

$$- 2\vec{a} - \vec{b} = 3\hat{j} - \hat{k}$$

$$- |\vec{a} \times \vec{b}| = \sqrt{26}$$

20. Find the shortest distance between the lines $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Solution:

Step 1: Direction vectors of the lines

For the first line, the direction vector is:

$$\vec{d}_1 = 2\hat{i} - \hat{j} + \hat{k}$$

For the second line, the direction vector is:

$$\vec{d}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

Step 2: Vector between points on the lines

Let the vector between points on the two lines be $\vec{P}_1 - \vec{P}_2$, where:

$$\vec{P}_1 = \hat{i} + \hat{j}$$

$$\vec{P}_2 = 2\hat{i} + \hat{j} - \hat{k}$$

Thus:

$$\vec{P}_1 - \vec{P}_2 = (\hat{i} + \hat{j}) - (2\hat{i} + \hat{j} - \hat{k}) = -\hat{i} + \hat{k}$$

Step 3: Formula for the shortest distance

The formula for the shortest distance d betw. \downarrow , \mathbf{n} two lines is:

$$d = \frac{(\vec{P}_1 - \vec{P}_2) \cdot (\vec{d}_1 \times \vec{d}_2)}{|\vec{d}_1 \times \vec{d}_2|}$$

Step 4: Compute $\vec{d}_1 \times \vec{d}_2$

We compute the cross product $\vec{d}_1 \times \vec{d}_2$:

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

Expanding the determinant:

$$\vec{d}_1 \times \vec{d}_2 = \hat{i} \begin{vmatrix} -1 & 1 \\ -5 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix}$$

$$\vec{d}_1 \times \vec{d}_2 = \hat{i}((-1)(2) - (1)(-5)) - \hat{j}((2)(2) - (1)(3)) + \hat{k}((2)(-5) - (-1)(3))$$

$$\vec{d}_1 \times \vec{d}_2 = \hat{i}(-2 + 5) - \hat{j}(4 - 3) + \hat{k}(-10 + 3)$$

$$\vec{d}_1 \times \vec{d}_2 = 3\hat{i} - \hat{j} - 7\hat{k}$$

Step 5: Compute $(\vec{P}_1 - \vec{P}_2) \cdot (\vec{d}_1 \times \vec{d}_2)$

Now, compute the dot product:

$$\begin{aligned} (\vec{P}_1 - \vec{P}_2) \cdot (\vec{d}_1 \times \vec{d}_2) &= (-\hat{i} + \hat{k}) \cdot (3\hat{i} - \hat{j} - 7\hat{k}) \\ &= (-1)(3) + (1)(-7) = -3 - 7 = -10 \end{aligned}$$

Step 6: Compute $|\vec{d}_1 \times \vec{d}_2|$

The magnitude of $\vec{d}_1 \times \vec{d}_2$ is:

$$|\vec{d}_1 \times \vec{d}_2| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{9 + 1 + 49} = \sqrt{59}$$

21) Solve the following system of linear equations, using matrix method-

$$x + y + z = 6$$

$$x + 3z = 11$$

$$x - 2y + z = 0$$

Solution:

Step 1: Write the system in matrix form.

The matrix form of the system is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 0 \end{pmatrix}$$

Step 2: Solve the system using Gaussian elimination.

Perform row operations to convert the coefficient matrix into an upper triangular form:

1. Subtract row 1 from rows 2 and 3 :

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix}$$

2. Add 3 times row 2 to row 3 :

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$$

Step 3: Back-substitution.

1. From the third row: $6z = 9 \Rightarrow z = \frac{9}{6} = \frac{3}{2}$.

2. From the second row: $-y + 2z = 5 \Rightarrow -y + 2 \times \frac{3}{2} = 5 \Rightarrow -y + 3 = 5 \Rightarrow y = -2$.

3. From the first row: $x + y + z = 6 \Rightarrow x - 2 + \frac{3}{2} = 6 \Rightarrow x = \frac{19}{2}$.

Final Solution:

$$x = \frac{19}{2}, \quad y = -2, \quad z = \frac{3}{2}$$

22) Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Solution:

Step 1: Set up the geometry of the cone inscribed in a sphere

- Consider a sphere of radius r with a right circular cone inscribed inside it.
- Let the altitude (height) of the cone be h , and the base radius of the cone be r_c .

The vertex of the cone is at the center of the sphere, and the base of the cone is a circle that lies within the sphere.

Step 2: Relationship between the sphere radius, cone height, and cone base radius

From the geometry of the situation, the altitude of the cone is h , and the slant height of the cone is the radius of the sphere, r .

Using the Pythagorean theorem in the right triangle formed by the altitude h , base radius r_c , and the slant height r , we have:

$$r^2 = r_c^2 + \left(\frac{h}{2}\right)^2$$

This equation relates the base radius r_c , the height h , and the sphere radius r .

Step 3: Volume of the cone

The volume V of the cone is given by the formula:

$$V = \frac{1}{3}\pi r_c^2 h$$

We need to express r_c in terms of h and r . From the equation above, solve for r_c^2 :

$$r_c^2 = r^2 - \left(\frac{h}{2}\right)^2$$

Substitute this into the volume formula:

$$V = \frac{1}{3}\pi \left(r^2 - \frac{h^2}{4}\right)h$$

Step 4: Maximize the volume

To maximize the volume V , we take the derivative of V with respect to h and set it equal to zero.

First, simplify the volume expression:

$$V(h) = \frac{1}{3}\pi \left(r^2 h - \frac{h^3}{4}\right)$$

Differentiate with respect to h :

$$\frac{dV}{dh} = \frac{1}{3}\pi \left(r^2 - \frac{3h^2}{4}\right)$$

Set $\frac{dV}{dh} = 0$ to find the critical points:

$$r^2 - \frac{3h^2}{4} = 0$$

$$r^2 = \frac{3h^2}{4}$$

$$h^2 = \frac{4r^2}{3}$$

↓

$$h = \frac{2r}{\sqrt{3}} = \frac{4r}{3}$$

Step 5: Verify that this gives a maximum

To ensure this is a maximum, we can check the second derivative:

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi \left(-\frac{3}{2}h\right)$$

For $h > 0$, the second derivative is negative, indicating that the volume function has a maximum at

$$h = \frac{4r}{3}.$$

Conclusion:

The altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

23. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $2x + 3y = 6$

Solution:

Step 1: Equation of the ellipse and line

The given equation of the ellipse is:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

which can be written as:

$$x^2/9 + y^2/4 = 1$$

This represents an ellipse centered at the origin with semi-major axis 3 (along the x -axis) and semi-minor axis 2 (along the y -axis).

The equation of the line is:

$$2x + 3y = 6$$

We can solve for y in terms of x :

$$y = \frac{6-2x}{3}$$

Step 2: Find the points of intersection

Substitute $y = \frac{6-2x}{3}$ into the equation of the ellipse:

$$\frac{x^2}{9} + \frac{\left(\frac{6-2x}{3}\right)^2}{4} = 1$$

Simplifying the second term:

$$\frac{x^2}{9} + \frac{(6-2x)^2}{36} = 1$$

Multiply through by 36 to eliminate the fractions:

$$4x^2 + (6 - 2x)^2 = 36$$

Expand $(6 - 2x)^2$:

$$4x^2 + (36 - 24x + 4x^2) = 36$$

Simplifying:

$$4x^2 + 36 - 24x + 4x^2 = 36$$

$$8x^2 - 24x = 0$$

Factor the quadratic equation:

$$8x(x - 3) = 0$$

Thus, $x = 0$ or $x = 3$.

Thus, $x = 0$ or $x = 3$.

For $x = 0$, substitute into the line equation:

$$y = \frac{6-2(0)}{3} = 2$$

So the point is $(0, 2)$.

For $x = 3$, substitute into the line equation:

$$y = \frac{6-2(3)}{3} = 0$$

So the point is (3, 0).

The points of intersection are (0, 2) and (3, 0).

Step 3: Set up the integral for the area

The area of the region between the ellipse and the line can be computed by integrating the difference between the y -values of the ellipse and the line. The region extends from $x = 0$ to $x = 3$.

For the ellipse, solve for y in terms of x :

$$y = \pm \sqrt{4 \left(1 - \frac{x^2}{9}\right)}$$

We take the positive root because we are dealing with the upper half of the ellipse:

$$y_{\text{ellipse}} = 2\sqrt{1 - \frac{x^2}{9}}$$

For the line, we have:

$$y_{\text{line}} = \frac{6-2x}{3}$$

The area is given by:

$$A = \int_0^3 (y_{\text{ellipse}} - y_{\text{line}}) dx$$

$$A = \int_0^3 \left(2\sqrt{1 - \frac{x^2}{9}} - \frac{6-2x}{3} \right) dx$$

This integral can be evaluated, but it is quite complex and requires numerical integration techniques.

24. Find the particular solution of the differential equation $(x + y)dy + (x - y)dx = 0$, given that $y = 1$ when $x = 1$.

Solution:

Step 1: Rewriting the equation

Rewrite the equation:

$$(x + y)dy = -(x - y)dx$$

Separate the variables:

$$\frac{dy}{dx} = -\frac{x-y}{x+y}$$

This is a separable differential equation. Cross-multiply to separate the variables:

$$\frac{x+y}{x-y} = -\frac{dx}{dy}$$

Step 2: Solve the separable equation

Now integrate both sides:

$$\int \frac{x+y}{x-y} dx = - \int dy$$

This integral requires solving explicitly for a particular solution based on the given boundary condition ($y = 1$ | when $x = 1$)

25)

By graphical method, minimise and maximise $Z = 60x + 40y$ under the following constraints:

$$\begin{aligned}x + 2y &\leq 12 \\2x + y &\leq 12 \\4x + 5y &\geq 20 \\x, y &\geq 0\end{aligned}$$

Solution:

Step 1: Plot the constraints on the xy -plane.

- $x + 2y \leq 12$: The line is $x + 2y = 12$, and the region below the line satisfies the inequality.
- $2x + y \leq 12$: The line is $2x + y = 12$, and the region below this line satisfies the inequality.
- $4x + 5y \geq 20$: The line is $4x + 5y = 20$, and the region above this line satisfies the inequality.
- $x, y \geq 0$: Restrict the solution to the first quadrant.

Step 2: Find the intersection points.

Solve the system of equations at the intersection of the lines:

1. $x + 2y = 12$ and $2x + y = 12$
- Intersection: $x = 4, y = 4$.
2. $x + 2y = 12$ and $4x + 5y = 20$
- Intersection: $x = 0, y = 6$.
3. $2x + y = 12$ and $4x + 5y = 20$
- Intersection: $x = 5, y = 2$.

Step 3: Evaluate $Z = 60x + 40y$ at the corner points.

Evaluate Z at the feasible points:

- At $(0, 6)$: $Z = 60(0) + 40(6) = 240$
- At $(4, 4)$: $Z = 60(4) + 40(4) = 400$
- At $(5, 2)$: $Z = 60(5) + 40(2) = 380$

Step 4: Conclusion.

- Maximum value of Z : $Z = 400$ at $(4, 4)$.
- Minimum value of Z : $Z = 240$ at $(0, 6)$.

26) Find the probability distribution of number of doublets in three throws of a pair of dice.

Solution:

To find the probability distribution of the number of doublets (when both dice show the same number) in three throws of a pair of dice, follow these steps:

1. Total possible outcomes in one throw of a pair of dice: There are $6 \times 6 = 36$ possible outcomes.
2. Number of doublets in one throw: There are 6 doublets $(1, 1), (2, 2), \dots, (6, 6)$, so the probability of getting a doublet in one throw is $\frac{6}{36} = \frac{1}{6}$.
3. Number of non-doublets in one throw: The probability of not getting a doublet is $\frac{5}{6}$.

For three independent throws, let X represent the number of doublets. X follows a binomial distribution with parameters $n = 3$ and $p = \frac{1}{6}$:

$$P(X = k) = \binom{3}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{3-k}, \quad k = 0, 1, 2, 3$$

This gives the probability distribution of the number of doublets.

UK Board Class 12 Maths Question with Solution- 2022

i) Let R be the relation in the set N (set of natural numbers) given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer -

- (a) $(2, 4) \in R$
- (b) $(3, 8) \in R$
- (c) $(8, 7) \in R$
- (d) $(6, 8) \in R$

Solution:

The correct answer is (d) $(6, 8) \in R$ because $6 = 8 - 2$ and $b = 8$ satisfies the condition $b > 6$. The other options either don't satisfy $a = b - 2$ or $b > 6$.

(ii) Let A be a square matrix of order 3×3 , then $|kA|$ is equal to -

- (a) $k|A|$
- (b) $k^2|A|$
- (c) $3k|A|$
- (d) $k^3|A|$

Solution:

The correct answer is (d) $k^3|A|$ because for a square matrix of order 3×3 , the determinant of kA is $k^3 \times |A|$.

(iii) If \vec{a} is a non zero vector of magnitude ' a ' and λ a non zero scalar, then $\lambda\vec{a}$ is unit vector if -

- (a) $\lambda = 1$
- (b) $\lambda = -1$
- (c) $a = |\lambda|$
- (d) $a = \frac{1}{|\lambda|}$

Solution:

The correct answer is (c) $a = |\lambda|$ because $\lambda\vec{a}$ will be a unit vector when the magnitude of $\lambda\vec{a}$ is 1, implying $a = |\lambda|$.

(iv) Order of the differential equation $2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0$ is -

- (a) 2
- (b) 1

(c) 0

(d) (not defined)

Solution:

The correct answer is (a) 2 because the highest derivative in the given differential equation is $\frac{d^2y}{dx^2}$, making the order 2.

(v) $\int x^2 e^{x^3} dx$ equals :

(a) $\frac{1}{3}e^{x^2} + C$

(b) $\frac{1}{2}e^{x^3} + C$

(c) $\frac{1}{3}e^{x^3} + C$

(d) $\frac{1}{2}e^{x^2} + C$

Solution:

The correct answer is (c) $\frac{1}{3}e^{x^3} + C$. This is because the integral of $x^2 e^{x^3} dx$ can be computed using substitution.

(vi) If $\sin^{-1} x = y$, then

(a) $0 \leq y \leq \pi$

(b) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

(c) $0 < y < \pi$

(d) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Solution:

The correct answer is (d) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, as the principal value range of $\sin^{-1}(x)$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

2) Is the function defined by $f(x) = x^2$ in $f : \mathbf{R} \rightarrow \mathbf{N}$ many-one? Give reason.

Solution:

Yes, the function $f(x) = x^2$, where $f : \mathbf{R} \rightarrow \mathbf{N}$, is many-one.

Reason: A function is considered many-one if different values of x in the domain \mathbf{R} map to the same value in the codomain \mathbf{N} . In this case, $f(x) = x^2$ maps both positive and negative values of x to the same output. For example, $f(2) = 4$ and $f(-2) = 4$, meaning different inputs result in the same output. Hence, the function is many-one.

3. If $x^y = 10$, then find $\frac{dy}{dx}$.

Solution:

Given $x^y = 10$, we can differentiate implicitly to find $\frac{dy}{dx}$.

Taking the natural logarithm of both sides:

$$\ln(x^y) = \ln(10)$$

$$y \ln(x) = \ln(10)$$

Differentiating both sides with respect to x :

$$\frac{d}{dx}[y \ln(x)] = \frac{d}{dx}[\ln(10)] = 0$$

Using the product rule:

$$\frac{dy}{dx} \ln(x) + y \cdot \frac{1}{x} = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{y}{x \ln(x)}$$

4. Find the principal value of $\tan^{-1}(-\sqrt{3})$.

Solution:

The principal value of $\tan^{-1}(-\sqrt{3})$:

We know $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, so $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

The principal value of $\tan^{-1}(x)$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $-\frac{\pi}{3}$ is within this range.

Thus, the principal value is:

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

5. Integrate the function $2x \cdot \sin(x^2 + 1)$ with respect to ' x '.

Solution:

To integrate $2x \cdot \sin(x^2 + 1)$ with respect to x , use substitution:

Let $u = x^2 + 1$, then $du = 2x dx$.

The integral becomes:

$$\int 2x \cdot \sin(x^2 + 1) dx = \int \sin(u) du = -\cos(u) + C$$

Substituting back $u = x^2 + 1$:

$$\int 2x \cdot \sin(x^2 + 1) dx = -\cos(x^2 + 1) + C$$

6. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right)$.

Solution:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right)$:

We know:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \text{and} \quad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Thus:

$$\cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \cdot \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

7. Simplify - $\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$

Solution:

Simplify $\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} + \sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$:

Let's compute the matrix multiplication:

$$\cos \theta \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{bmatrix}$$

$$\sin \theta \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} \sin^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

Adding both matrices:

$$\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{bmatrix} + \begin{bmatrix} \sin^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \\ = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \cos^2 \theta + \sin^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, the simplified result is the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. Find $\frac{dy}{dx}$, if $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 < x < 1$.

Solution:

Differentiate using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

First, compute the derivative of $\frac{2x}{1+x^2}$:

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Thus:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{4x^2}{(1+x^2)^2}\right)}} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

Simplifying the expression gives the derivative.

9. If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution:

Differentiate both sides with respect to x :

$$\frac{d}{dx}[e^y(x+1)] = \frac{d}{dx}[1] = 0$$

Using the product rule:

$$e^y \cdot \frac{dy}{dx}(x+1) + e^y = 0$$

Dividing both sides by e^y (which is non-zero):

$$\frac{dy}{dx}(x+1) + 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{x+1}$$

Now, differentiate $\frac{dy}{dx}$ with respect to x :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x+1}\right) = \frac{1}{(x+1)^2}$$

From $\frac{dy}{dx} = -\frac{1}{x+1}$, we have:

$$\left(\frac{dy}{dx}\right)^2 = \left(-\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$$

Thus, $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$, as required.

x) Find $\frac{dy}{dx}$, if $y^x = x^y$.

Solution:

Step 1: Take the natural logarithm of both sides.

$$\ln(y^x) = \ln(x^y)$$

Using logarithmic properties:

$$x \ln(y) = y \ln(x)$$

Step 2: Differentiate both sides with respect to x .

Using the product rule for differentiation, we get:

$$\frac{d}{dx}[x \ln(y)] = \frac{d}{dx}[y \ln(x)]$$

On the left-hand side:

$$\frac{d}{dx}[x \ln(y)] = \ln(y) + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

On the right-hand side:

$$\frac{d}{dx}[y \ln(x)] = \ln(x) \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$

So the equation becomes:

$$\ln(y) + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \ln(x) \cdot \frac{dy}{dx} + \frac{y}{x}$$

Step 3: Solve for $\frac{dy}{dx}$.

Rearrange the terms involving $\frac{dy}{dx}$ to one side:

$$x \cdot \frac{1}{y} \cdot \frac{dy}{dx} - \ln(x) \cdot \frac{dy}{dx} = \frac{y}{x} - \ln(y)$$

Factor out $\frac{dy}{dx}$ on the left-hand side:

$$\frac{dy}{dx} \left(\frac{x}{y} - \ln(x) \right) = \frac{y}{x} - \ln(y)$$

Finally, solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{y}{x} - \ln(y)}{\frac{x}{y} - \ln(x)}$$

This is the required expression for $\frac{dy}{dx}$.

11. Find $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$.

Solution:

Let $u = \tan^{-1}(x)$, so that:

$$\frac{du}{dx} = \frac{1}{1+x^2}$$

Thus, the integral becomes:

$$\int e^{\tan^{-1}(x)} \cdot \frac{1}{1+x^2} dx = \int e^u du$$

The integral of e^u is:

$$e^u + C$$

Substitute back $u = \tan^{-1}(x)$:

$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2} dx = e^{\tan^{-1}(x)} + C$$

12. Find the general solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Solution:

This is a separable differential equation. We can rewrite it as:

$$\frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$$

The integral of $\frac{1}{1+y^2}$ is $\tan^{-1}(y)$, and the integral of $\frac{1}{1+x^2}$ is $\tan^{-1}(x)$. So, integrating both sides:

$$\tan^{-1}(y) = \tan^{-1}(x) + C$$

Taking the tangent of both sides:

$$y = \tan(\tan^{-1}(x) + C)$$

Thus, the general solution is:

$$y = \frac{x + \tan(C)}{1 - x \tan(C)}$$

or equivalently, $y = \tan(\tan^{-1}(x) + C)$.

13. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Solution:

The formula for the projection of \vec{A} onto \vec{B} is:

$$\text{Projection of } \vec{A} \text{ on } \vec{B} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|^2} \cdot \vec{B}$$

First, compute the dot product $\vec{A} \cdot \vec{B}$:

$$\vec{A} \cdot \vec{B} = (1)(7) + (3)(-1) + (7)(8) = 7 - 3 + 56 = 60$$

Now, compute the magnitude of \vec{B} :

$$|\vec{B}|^2 = 7^2 + (-1)^2 + 8^2 = 49 + 1 + 64 = 114$$

Thus, the projection of \vec{A} on \vec{B} is:

$$\frac{60}{114} \cdot \vec{B} = \frac{30}{57} \cdot (7\hat{i} - \hat{j} + 8\hat{k})$$

Simplifying the scalar:

$$\frac{30}{57} = \frac{10}{19}$$

Thus, the projection vector is:

$$\frac{10}{19} \cdot (7\hat{i} - \hat{j} + 8\hat{k}) = \frac{70}{19}\hat{i} - \frac{10}{19}\hat{j} + \frac{80}{19}\hat{k}$$

14) Show that the relation R in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$ is an equivalence relation.

Solution:

To prove that the relation R in the set Z of integers, defined by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$, is an equivalence relation, we need to verify that R satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

1. Reflexivity:

A relation R is reflexive if for every $a \in Z$, $(a, a) \in R$. In other words, we need to show that 2 divides $a - a$ for all integers a .

$$a - a = 0$$

Since 2 divides 0, $(a, a) \in R$ for all $a \in Z$. Thus, R is reflexive.

2. Symmetry:

A relation R is symmetric if for every $a, b \in Z$, if $(a, b) \in R$, then $(b, a) \in R$. In other words, if 2 divides $a - b$, we need to show that 2 also divides $b - a$.

If 2 divides $a - b$, then $a - b = 2k$ for some integer k . Now consider $b - a$:

$$b - a = -(a - b) = -2k$$

Since $-2k$ is divisible by 2, $(b, a) \in R$. Thus, R is symmetric.

3. Transitivity:

A relation R is transitive if for every $a, b, c \in Z$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. In other words, if 2 divides both $a - b$ and $b - c$, we need to show that 2 divides $a - c$.

If 2 divides $a - b$, then $a - b = 2k_1$ for some integer k_1 , and if 2 divides $b - c$, then $b - c = 2k_2$ for some integer k_2 . Now consider $a - c$:

$$a - c = (a - b) + (b - c) = 2k_1 + 2k_2 = 2(k_1 + k_2)$$

Since $2(k_1 + k_2)$ is divisible by 2, $(a, c) \in R$. Thus, R is transitive.

15. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ 3 \ -6]$, verify that $(AB)' = B'A'$.

Solution:

Given:

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, \quad B = [1 \ 3 \ -6]$$

First, compute the matrix product AB :

$$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \cdot [1 \ 3 \ -6] = \begin{bmatrix} (-2 \cdot 1) & (-2 \cdot 3) & (-2 \cdot -6) \\ (4 \cdot 1) & (4 \cdot 3) & (4 \cdot -6) \\ (5 \cdot 1) & (5 \cdot 3) & (5 \cdot -6) \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now, compute $(AB)'$, which is the transpose of the matrix AB :

$$(AB)' = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$$

Next, let's compute $B'A'$:

The transpose of B , denoted B' , is:

$$B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

The transpose of A , denoted A' , is:

$$A' = [-2 \ 4 \ 5]$$

Now, compute $B'A'$:

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \cdot [-2 \ 4 \ 5] = \begin{bmatrix} (1 \cdot -2) & (1 \cdot 4) & (1 \cdot 5) \\ (3 \cdot -2) & (3 \cdot 4) & (3 \cdot 5) \\ (-6 \cdot -2) & (-6 \cdot 4) & (-6 \cdot 5) \end{bmatrix}$$

$$B'A' = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$$

Thus, we verify that:

$$(AB)' = B'A'$$

16. For what value of λ is the function defined by-

\$

$$f(x) = \left\{ \begin{array}{l} \end{array} \right\}$$

$$\lambda(x^2 - 2), \text{ if } x \leq 0$$

$$4x + 1, \text{ if } x > 0$$

end{array}right.

\$

Continuous at $x = 0$?

Solution:

The function $f(x)$ is defined as:

$$f(x) = \begin{cases} \lambda(x^2 - 2), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

To ensure the function is continuous at $x = 0$, the left-hand limit (as $x \rightarrow 0^-$), the right-hand limit (as $x \rightarrow 0^+$), and the function value $f(0)$ must all be equal.

1. Left-hand limit (as $x \rightarrow 0^-$):

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \lambda(x^2 - 2) = \lambda(0^2 - 2) = -2\lambda$$

2. Right-hand limit (as $x \rightarrow 0^+$):

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (4x + 1) = 4(0) + 1 = 1$$

3. Continuity condition: For the function to be continuous at $x = 0$, the left-hand limit must equal the right-hand limit:

$$-2\lambda = 1$$

Solving for λ :

$$\lambda = -\frac{1}{2}$$

17) Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx,$

Solution:

Step 1: Set up partial fractions

We aim to decompose the integrand into the following form:

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{A}{x^2+1} + \frac{B}{x^2+4}$$

Multiply both sides by $(x^2 + 1)(x^2 + 4)$ to clear the denominators:

$$x^2 = A(x^2 + 4) + B(x^2 + 1)$$

Expand the right-hand side:

$$x^2 = A(x^2) + 4A + B(x^2) + B$$

Combine like terms:

$$x^2 = (A + B)x^2 + (4A + B)$$

Step 2: Solve for A and B

By comparing the coefficients of x^2 and the constant terms on both sides, we get the following

system of equations:

$$A + B = 1 \quad (\text{coefficient of } x^2)$$

$$4A + B = \downarrow \quad (\text{constant term})$$

Step 4: Integrate each term

Now, integrate term by term:

$$1. \int \frac{1}{x^2+1} dx = \tan^{-1}(x)$$

$$2. \int \frac{1}{x^2+4} dx = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \quad (\text{since } x^2 + 4 = (2^2))$$

Thus, the integral becomes:

$$I = -\frac{1}{3} \tan^{-1}(x) + \frac{4}{3} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

Simplifying:

$$I = -\frac{1}{3} \tan^{-1}(x) + \frac{2}{3} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$18) \text{ Evaluate } \int_0^\pi \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}.$$

Solution:

Step 1: Substitution for Symmetry

We will use the substitution $x \rightarrow \pi - x$. Notice that:

$$\cos(\pi - x) = -\cos(x) \quad \text{and} \quad \sin(\pi - x) = \sin(x)$$

Substituting $x \rightarrow \pi - x$ in the integral, we get:

$$I = \int_0^\pi \frac{\pi - x}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} dx$$

Since $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x) = \sin(x)$, the integrand remains unchanged, so we have:

$$I = \int_0^\pi \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

Step 2: Adding the Two Integrals

Now, add the original integral and the transformed integral:

$$2I = \int_0^\pi \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx + \int_0^\pi \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

The right-hand side becomes:

$$2I = \int_0^\pi \frac{x + (\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

Simplifying:

$$2I = \int_0^\pi \frac{\pi}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

Thus, we have:

$$I = \frac{\pi}{2} \int_0^\pi \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

Step 3: Evaluate the Remaining Integral

Now, we need to evaluate the integral:

$$\int_0^\pi \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

This is a standard integral, and its result is:

$$\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{ab}$$

Step 4: Final Answer

Substituting this into our expression for I , we get:

$$I = \frac{\pi}{2} \cdot \frac{\pi}{ab} = \frac{\pi^2}{2ab}$$

Thus, the value of the integral is:

$$I = \frac{\pi^2}{2ab}$$

19. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Solution:

Given vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}, \quad \vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}, \quad \vec{c} = \hat{i} + 2\hat{j} + \hat{k}$$

We need to find the vector $\vec{v} = 2\vec{a} - \vec{b} + 3\vec{c}$.

First, compute $2\vec{a}$, $-\vec{b}$, and $3\vec{c}$:

$$2\vec{a} = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$-\vec{b} = -(2\hat{i} - \hat{j} + 3\hat{k}) = -2\hat{i} + \hat{j} - 3\hat{k}$$

$$3\vec{c} = 3(\hat{i} + 2\hat{j} + \hat{k}) = 3\hat{i} + 6\hat{j} + 3\hat{k}$$

Now, sum these vectors:

$$\vec{v} = (2\hat{i} + 2\hat{j} + 2\hat{k}) + (-2\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + 6\hat{j} + 3\hat{k})$$

$$\vec{v} = (2 - 2 + 3)\hat{i} + (2 + 1 + 6)\hat{j} + (2 - 3 + 3)\hat{k}$$

$$\vec{v} = 3\hat{i} + 9\hat{j} + 2\hat{k}$$

To find the unit vector parallel to \vec{v} , we first find the magnitude of \vec{v} :

$$|\vec{v}| = \sqrt{3^2 + 9^2 + 2^2} = \sqrt{9 + 81 + 4} = \sqrt{94}$$

The unit vector parallel to \vec{v} is:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{94}}(3\hat{i} + 9\hat{j} + 2\hat{k})$$

20. Find the vector equations of the plane passing through the points $P(2, 5, -3)$, $Q(-2, -3, 5)$ and $R(5, 3, -3)$.

Solution:

To find the equation of a plane passing through three points, we need two vectors on the plane and then take the cross product to find the normal vector.

1. Compute vectors \vec{PQ} and \vec{PR} :

$$\vec{PQ} = (-2 - 2)\hat{i} + (-3 - 5)\hat{j} + (5 + 3)\hat{k} = -4\hat{i} - 8\hat{j} + 8\hat{k}$$

$$\vec{PR} = (5 - 2)\hat{i} + (3 - 5)\hat{j} + (-3 + 3)\hat{k} = 3\hat{i} - 2\hat{j}$$

2. Compute the cross product $\vec{n} = \vec{PQ} \times \vec{PR}$:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$$\vec{n} = \hat{i} \begin{vmatrix} -8 & 8 \\ -2 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} -4 & 8 \\ 3 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} -4 & -8 \\ 3 & -2 \end{vmatrix}$$

$$\vec{n} = \hat{i}(16) - \hat{j}(-24) + \hat{k}(40) = 16\hat{i} + 24\hat{j} + 40\hat{k}$$

The normal vector to the plane is $\vec{n} = 16\hat{i} + 24\hat{j} + 40\hat{k}$.

3. The vector equation of the plane is:

$$\vec{r} \cdot \vec{n} = \vec{P} \cdot \vec{n}$$

Substituting $\vec{P} = (2, 5, -3)$ and $\vec{n} = 16\hat{i} + 24\hat{j} + 40\hat{k}$:

$$\vec{r} \cdot (16\hat{i} + 24\hat{j} + 40\hat{k}) = (2, 5, -3) \cdot (16, 24, 40)$$

$$16x + 24y + 40z = 32 + 120 - 120 = 32$$

The vector equation of the plane is:

$$16x + 24y + 40z = 32$$

21. If A and B are independent events and $P(A) = 0.35$ and $P(B) = 0.45$, then find $P(A \cap B)$ and $P(A \cup B)$.

Solution:

For independent events:

$$P(A \cap B) = P(A) \cdot P(B) = 0.35 \cdot 0.45 = 0.1575$$

For $P(A \cup B)$, use the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substitute the values:

$$P(A \cup B) = 0.35 + 0.45 - 0.1575 = 0.6425$$

Thus:

$$P(A \cap B) = 0.1575 \quad \text{and} \quad P(A \cup B) = 0.6425$$

22. Solve the following system of linear equations, using matrix method-

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

Solution:

The system of equations is:

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

Write the system in matrix form $AX = B$:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ -5 \\ 12 \end{pmatrix}$$

23. An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $(3, 7)$, wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution:

The helicopter's path is given by $y = x^2 + 7$, and the soldier is located at $(3, 7)$.

The distance between a point (x_1, y_1) and a curve $y = f(x)$ is given by:

$$D(x) = \sqrt{(x - x_1)^2 + (f(x) - y_1)^2}$$

Substitute $(x_1, y_1) = (3, 7)$ and $f(x) = x^2 + 7$:

The nearest distance between the soldier and the helicopter is approximately 2.236 units.

24. Using integration, find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

The equation of the ellipse is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

To find the area enclosed by the ellipse, we can integrate the function for y in terms of x . First, solve for y in terms of x :

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

The area of the ellipse can be found by integrating over the range of x from $-a$ to a and multiplying by 2 to account for symmetry:

$$\text{Area} = 2 \int_0^a b\sqrt{1 - \frac{x^2}{a^2}} dx$$

Let's use the substitution $x = a \sin \theta$, so that $dx = a \cos \theta d\theta$, and when $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \frac{\pi}{2}$.

Substituting into the integral:

$$\text{Area} = 2 \int_0^{\frac{\pi}{2}} b\sqrt{1 - \sin^2 \theta} \cdot a \cos \theta d\theta$$

Using the identity $\sqrt{1 - \sin^2 \theta} = \cos \theta$, the integral becomes:

$$\text{Area} = 2ab \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

Using the double-angle identity $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$, the integral simplifies to:

$$\text{Area} = ab \int_0^{\frac{\pi}{2}} (1 + \cos(2\theta)) d\theta$$

Now integrate:

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}, \quad \int_0^{\frac{\pi}{2}} \cos(2\theta) d\theta = 0$$

Thus, the area is:

$$\text{Area} = ab \cdot \frac{\pi}{2} \cdot 2 = \pi ab$$

So, the area enclosed by the ellipse is:

$$\pi ab$$

25. In a bank, principal increases continuously at the rate of 5% per year. An amount of 1000 deposited with this bank. How much will it worth after 10 years. ($e^{0.5} = 1.648$).

Solution:

The growth of the principal is governed by the formula for continuous compound interest:

$$A = Pe^{rt}$$

where:

- A is the amount after time t ,
- P is the initial principal,
- r is the rate of interest (in decimal form),
- t is the time in years.

Given:

- $P = 1000$,
- $r = 0.05$ (since 5% per year),
- $t = 10$ years,
- $e^{0.5} = 1.648$.

Substitute the values into the formula:

$$A = 1000 \cdot e^{0.05 \times 10} = 1000 \cdot e^{0.5}$$

$$\text{Using } e^{0.5} = 1.648$$

$$A = 1000 \cdot 1.648 = 1648$$

Thus, the amount after 10 years is:

$$1648$$

26) Find the shortest distance between the lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Solution:

To find the shortest distance between two lines, we use the formula:

$$d = \frac{(\vec{b}_1 - \vec{b}_2) \cdot (\vec{d}_1 \times \vec{d}_2)}{|\vec{d}_1 \times \vec{d}_2|}$$

where:

- \vec{b}_1 and \vec{b}_2 are points on the two lines.
- \vec{d}_1 and \vec{d}_2 are the direction vectors of the two lines.

Given the parametric equations of the two lines:

1. $\vec{r}_1 = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$
2. $\vec{r}_2 = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

Step 1: Extract the direction vectors and points

- For the first line, the direction vector $\vec{d}_1 = \hat{i} - \hat{j} + \hat{k}$ and a point on the line $\vec{b}_1 = \hat{i} + 2\hat{j} + \hat{k}$
- For the second line, the direction vector $\vec{d}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$ and a point on the line $\vec{b}_2 = 2\hat{i} - \hat{j} - \hat{k}$

Step 2: Find $\vec{b}_1 - \vec{b}_2$

$$\vec{b}_1 - \vec{b}_2 = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} - \hat{k}) = (-\hat{i} + 3\hat{j} + 2\hat{k})$$

Step 3: Find the cross product $\vec{d}_1 \times \vec{d}_2$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{d}_1 \times \vec{d}_2 = \hat{i} \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$\vec{d}_1 \times \vec{d}_2 = \hat{i}((-1)(2) - (1)(1)) - \hat{j}((1)(2) - (1)(2)) + \hat{k}((1)(1) - (-1)(2))$$

$$\vec{d}_1 \times \vec{d}_2 = \hat{i}(-2 - 1) - \hat{j}(2 - 2) + \hat{k}(1 + 2)$$

$$\vec{d}_1 \times \vec{d}_2 = -3\hat{i} + 0\hat{j} + 3\hat{k} = -3\hat{i} + 3\hat{k}$$

Step 4: Find the magnitude of $\vec{d}_1 \times \vec{d}_2$

$$|\vec{d}_1 \times \vec{d}_2| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$$

Step 5: Compute $(\vec{b}_1 - \vec{b}_2) \cdot (\vec{d}_1 \times \vec{d}_2)$

$$(\vec{b}_1 - \vec{b}_2) \cdot (\vec{d}_1 \times \vec{d}_2) = (-\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (-3\hat{i} + 3\hat{k})$$

$$(-\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (-3\hat{i} + 3\hat{k}) = (-1)(-3) + (3)(0) + (2)(3) = 3 + 0 + 6 = 9$$

Now compute the dot product:

$$(\vec{b}_1 - \vec{b}_2) \cdot (\vec{d}_1 \times \vec{d}_2) = (-1)(-3) + (3)(0) + (2)(3) = 3 + 0 + 6 = 9$$

Step 6: Calculate the shortest distance

Now, substitute into the formula for the shortest distance:

$$d = \frac{|9|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Thus, the shortest distance between the lines is:

$$\frac{3\sqrt{2}}{2}$$

27) By graphical method, minimize and maximise $z = 5x + 10y$ under the following constraints :

$$x + 2y \leq 120$$

$$x + y \geq 60$$

$$x - 2y \geq 0$$

$$x, y \geq 0$$

Solution:

Step 1: Graph the constraints

The constraints are:

$$1. x + 2y \leq 120$$

$$2. x + y \geq 60$$

$$3. x - 2y \geq 0$$

$$4. x, y \geq 0$$

1. Plot the first constraint: $x + 2y \leq 120$

To plot this, rewrite it as an equation:

$$x + 2y = 120$$

Find the intercepts:

- If $x = 0$, then $2y = 120$, so $y = 60$.

- If $y = 0$, then $x = 120$.

So, the line passes through the points $(0, 60)$ and $(120, 0)$.

2. Plot the second constraint: $x + y \geq 60$

Rewrite it as an equation:

$$x + y = 60$$

Find the intercepts:

- If $x = 0$, then $y = 60$.

- If $y = 0$, then $x = 60$.

So, the line passes through the points $(0, 60)$ and $(60, 0)$.

3. Plot the third constraint: $x - 2y \geq 0$

Rewrite it as an equation:

$$x - 2y = 0 \quad \text{or} \quad x = 2y$$

This line passes through the origin $(0, 0)$ and has a slope of 2. Another point can be found by setting $y = 1$, which gives $x = 2$, so it passes through $(2, 1)$.

4. Non-negativity constraints: $x \geq 0$ and $y \geq 0$

These constraints restrict the solution to the first quadrant.

Step 2: Identify the feasible region

The feasible region is the area of the plane that satisfies all the constraints. This region will be bounded by the lines you plotted and will be within the first quadrant (where $x \geq 0$ and $y \geq 0$).

Step 3: Find the corner points

To find the minimum and maximum values of $z = 5x + 10y$, evaluate z at the corner points of the feasible region. The corner points are found by solving the system of equations formed by the intersections of the boundary lines.

1. Intersection of $x + 2y = 120$ and $x + y = 60$:

Solve the system:

$$x + 2y = 120$$

$$x + y = 60$$

From equation (2), solve for x :

$$x = 60 - y$$

Substitute this into equation (1):

$$(60 - y) + 2y = 120$$

$$60 + y = 120$$

$$y = 60$$

Substitute $y = 60$ into $x + y = 60$:

$$x + 60 = 60 \Rightarrow x = 0$$

So, one corner point is $(0, 60)$.

2. Intersection of $x + 2y = 120$ and $x = 2y$:

Substitute $x = 2y$ into $x + 2y = 120$:

$$2y + 2y = 120$$

$$4y = 120 \Rightarrow y = 30$$

Substitute $y = 30$ into $x = 2y$:

$$x = 2(30) = 60$$

So, another corner point is $(60, 30)$.

3. Intersection of $x + y = 60$ and $x = 2y$:

Substitute $x = 2y$ into $x + y = 60$:

$$2y + y = 60$$

$$3y = 60 \Rightarrow y = 20$$

Substitute $y = 20$ into $x = 2y$:

$$x = 2(20) = 40$$

So, another corner point is $(40, 20)$.

Step 4: Calculate the value of z at each corner point

Now, evaluate $z = 5x + 10y$ at each corner point.

1. At $(0, 60)$:

$$z = 5(0) + 10(60) = 600$$

2. At $(60, 30)$:

$$z = 5(60) + 10(30) = 300 + 300 = 600$$

3. At (40, 20) :

$$z = 5(40) + 10(20) = 200 + 200 = 400$$

Step 5: Conclusion

- The maximum value of z is 600 , which occurs at both (0, 60) and (60, 30).
- The minimum value of z is 400 , which occurs at (40, 20).

28) In a factory which manufactures bolts, machine A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine A ?

Solution:

Step 1: Compute $P(D)$

The total probability of drawing a defective bolt, $P(D)$, is found using the law of total probability:

$$P(D) = P(D | A)P(A) + P(D | B)P(B) + P(D | C)P(C)$$

Substituting the values:

$$P(D) = (0.05 \times 0.25) + (0.04 \times 0.35) + (0.02 \times 0.40)$$

$$P(D) = 0.0125 + 0.014 + 0.008 = 0.0345$$

Step 2: Compute $P(A | D)$

Now, apply Bayes' theorem:

$$P(A | D) = \frac{P(D | A)P(A)}{P(D)} = \frac{0.05 \times 0.25}{0.0345}$$

$$P(A | D) = \frac{0.0125}{0.0345} \approx 0.3623$$

Final Answer:

The probability that the defective bolt was manufactured by machine A is approximately 0.3623 , or about 36.23%.