CAREERS 360

PREPARATION Series

Class 11-12

Mathematics

Formula Book

Content

Class 11

Chapter 1: Sets

Set: A set is a well-defined collection of distinct objects and it is usually denoted by capital letters A, B, C, S, U, V……

Example: A = {1,2,3}

All the objects that form a set are called its elements or members. These are usually denoted by small letters, i.e. x, y, z …..

If x is an element of a set A, we write x ∈ A and read as ‘x belongs to A'.

If x is not an element of a set A, we write x ∉ A and read as ‘x does not belong to A'.

Example: A={1,2,3}, then 2 ∈ A (2 belongs to set A) and 4 ∉ A (4 does not belong to set A)

There are two methods of representing a set - Roster (or Tabular) form & Set-builder Form.

Roster or Tabular form

In roster form, all the elements of a set are listed, the elements are separated by commas and are enclosed within braces { }.

Set-builder Form

In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. If Z contains all values of x for which the condition q(x) is true, then we write

Z = { x : q(x) } or Z = { x | q(x) }

Where, ' : ' or ' | ' is read as 'such that'

eg. The set A = { 0, 1, 8, 27, 64, ... } is in Roster form, can be written in Set Builder form as

A = $\{x^3 : x$ is a non negative integer}

Empty Set

A set which does not contain any element in it is called the empty set (or null set or void set).

eg. A = {1<x<2, x is a natural number},

Since no natural number lies between 1 and 2, hence A will be an empty set.

The empty set is denoted by the symbol φ or { }.

Note: φ ≠ {φ}, φ ≠ {0}

Equal Sets

Two sets A and B are said to be equal if they have exactly the same elements and we write A = B.

Otherwise, the sets are said to be unequal and we write A ≠ B.

Example If A= {3,2,1,4} and B= {2,3,4,1}, then both have exactly same elements, and hence A = B.

Cardinal Number

The number of elements in a set is called its cardinal number. It is denoted by n(A).

If A = {a,s,d}, then n(A) = 3

and if B = {x : x^2 = 1}, then B = {1, -1 }, and hence n(B) = 2

Equivalent Sets

Two sets having the same number of elements are called equivalent sets.

Example: A= {H,T,P,V} and B= {1,2,3,4}, they both are equivalent as number of elements in both are same.

Equivalent sets have the same cardinal number

Note: Two equivalent sets may or may not be equal, but equal sets are always equivalent.

Finite set: A set which is empty or consists of a finite number of elements is called a finite set.

Examples: φ, {a}, {1,2,5,9}, {x: x is a person of age more than 18}

Infinite set: A set which has infinite elements is called an infinite set.

Examples:

a set of all the lines passing through a point,

set of all circles in a plane,

set of all points in a plane,

N, Z, Q, Q', R

{x : 2 < x < 2.1}

Singleton set: A set which is having only one element is called a singleton set.

Example: {3}, {b},

{{1,2,3}} is also singleton as it has one element which is a set

{φ} is also a singleton set

Note: Empty and singleton sets are finite sets

Subset

A set A is said to be a subset of a set B if all elements of A are present in B.

It is represented by ⊂ .

A = {1, 2, 3} ,   B = {1, 2, 3, 4, 5} ,   C = {3, 2, 1}

A is a subset of B or A ⊂ B

C is a subset of A or C ⊂ A

Also, if all elements of set A present in set B, then A is a **subset** of B and B is called a **superset** of A

Properties

1. If sets A and B are subsets of each other then they are equal sets.

   eg, If A = {1, 2, 3} , B = {3, 1, 2},

   Here, A ⊂ B and B ⊂ A ⇒ A = B

2. Every set is a subset of itself, A ⊂ A.

3. φ is a subset of every set.

Proper and Improper Subsets

If A is a subset of B but $A \neq B$, then we say A is a proper subset of B.

And if $A = B$, then we say that A is an improper subset of B.

Example: If A={2,4} and B= {1,2,3,4,5}, then A is a proper subset of B

And C = {1,2,3,4,5} is improper subset of B in this case

Note:

1. Every set has one improper subset, all other subsets are proper subsets.

2. φ has only one subset, which is φ itself. So, φ does not have any proper subset.

3. Important sets related to numbers

N : the set of all natural numbers

Z : the set of all integers

Q : the set of all rational numbers

Q': the set of all irrational numbers

R : the set of real numbers

Z^+ : the set of positive integers

Q^+ : the set of positive rational numbers

R^+ : the set of positive real numbers.

N ⊂ Z ⊂ Q ⊂ R

Number of subsets of a set

If a set A has n elements, then the total number of subsets of A is 2^n.

Also as each subset has one improper subset, so number of proper subsets is (2^n-1).

Intervals as Subset of R

Open Interval

If a and b are two real number such that a < b, then the set of all real numbers x such that

a < x < b is called an open interval and denoted by (a, b) or]a , b[.

(a, b) = {x ∈ R : a < x < b}.

On the real line, (a, b) represented as

Encircling a and b means the numbers a and b not included in the set.

So (1,5) contains all real numbers lying between 1 and 5, but NOT 1 and 5

So, 1.00001, 1.1, 1.5, π, √2, 4.99999 etc lie in this interval

Closed Interval

If a and b are two real number such that a < b, then the set of all real numbers x such that

a ≤ x ≤ b is called a closed interval and denoted by [a, b].

[a, b] = {x ∈ R : a ≤ x ≤ b}.

On the real line, [a, b] represented as

So [1,5] contains all real numbers lying between 1 and 5, including 1 and 5

So, 1.00001, 1.1, 1.5, π, √2, 4.99999 etc lie in this interval. Also 1 and 5 also lie in it

Open-Closed and Closed-open Intervals

If a and b are two real number such that a < b, then the set of all real numbers x such that

a ≤ x < b is called Closed-open interval and a < x ≤ b is called Open-closed interval.

These are denoted by [a, b) and (a, b] respectively

[a, b) = {x ∈ R : a ≤ x< b}: contains all real numbers between a and b, also it includes a (but does not include b)

(a, b] = {x ∈ R : a < x ≤ b}: contains all real numbers between a and b, also it includes b (but does not include a)

On the real line, these are represented as

[a, b)

(a, b]

Power set

The collection of all subsets of a set A is called the power set of A. It is denoted by P(A).

For Example if set A = {a, b, c}, then

P(A) = {φ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c}}

- The power set of any set is non-empty.
- Each element of a Power set is a set.
- For a set A, having n elements, the number of elements in power set $P(A)$ is given by: $2^n = \text{ no. of subsets in A}$

Universal set

A set that contains all sets in a given context is called the "Universal Set". The universal set is usually denoted by U, and all its subsets denoted by the letters A, B, C, etc.

For example, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set R of real numbers.

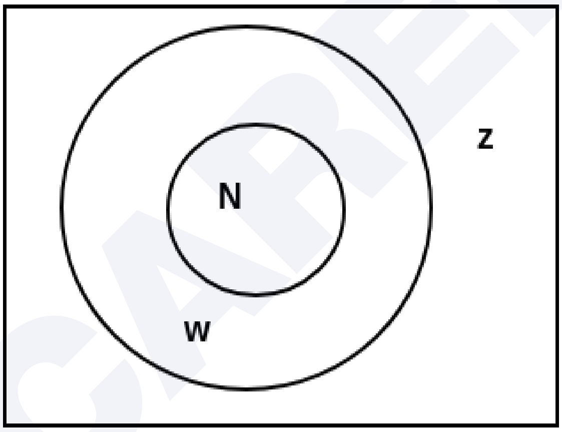
If A is set of all tigers in a jungle, and B is a set of all deers in the jungle, then universal set can be all the animals of that jungle, as all tigers and all deers are subsets of this set.

Venn Diagram

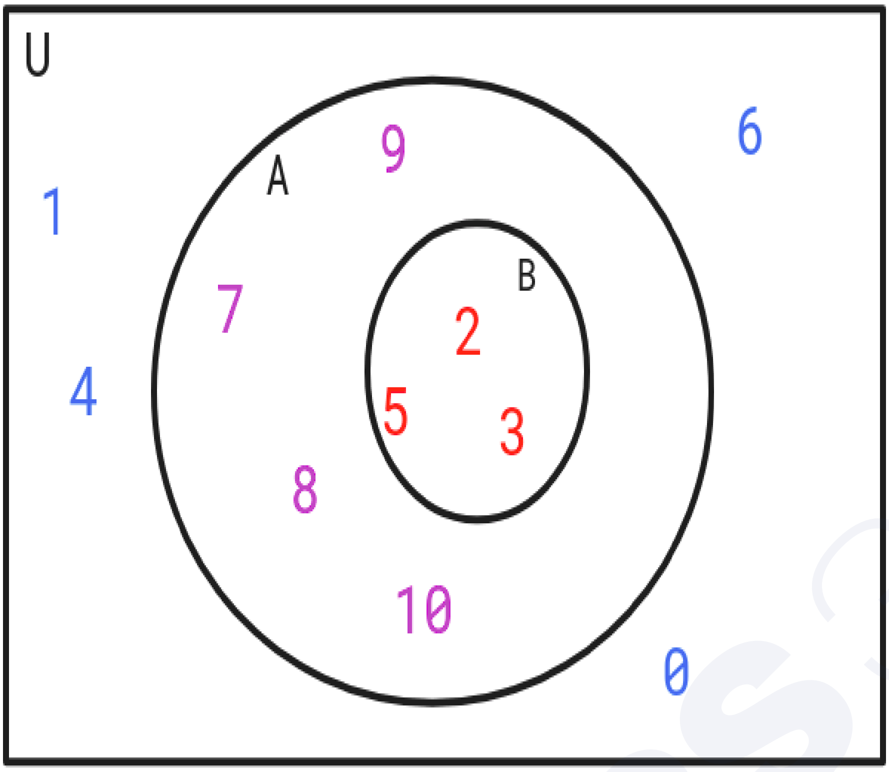
A diagram that represents or shows different sets is called a Venn diagram.

Universal set (U) is usually represented by a rectangle and its subsets are usually represented by circles (or any other closed curve).

For example, the set of natural numbers (N) is a subset of the set of whole numbers (W) which is a subset of integers (here integer(Z) is the universal set).



Example



In the above figure Universal Set (U) = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Set A = {2, 3, 5, 7, 8, 9, 10} and B = {2,3, 5}

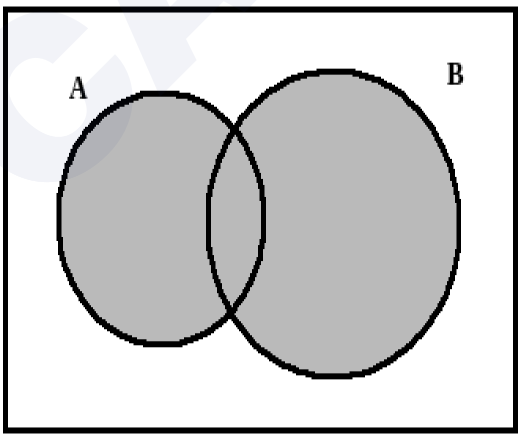
A is a subset of U (A ⊂ U)

B is a subset of A (B ⊂ A)

Union of Sets

Let A and B be any two sets. The union of A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '∪' is used to denote the union.

Symbolically, we write A U B = {x: x ∈ A or x ∈ B}.



AUB

Properties of union

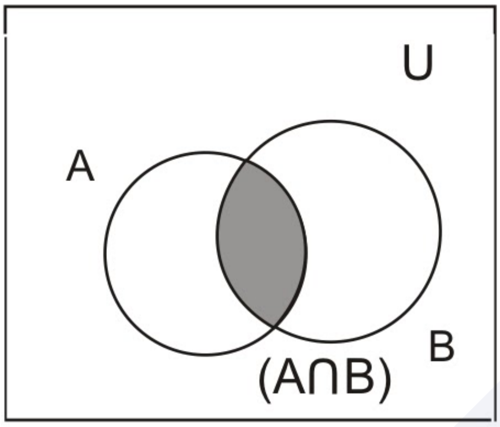
- A ∪ B = B ∪ A (Commutative Property)
- (A ⋃ B) ⋃ C = A ⋃ (B ⋃ C) (Associative property)
- A ⋃ φ = A (Law of identity element, φ is the identity of Null Set)
- A ⋃ A = A (Idempotent law)
- U ⋃ A = U (Law of U)
- If A is a subset of B, then A ∪ B = B

Intersection of sets

The intersection of sets A and B is the set of all elements which are common to both A and B. The symbol '∩ 'is used to denote the intersection.

Symbolically, we write A ∩ B = {x: x ∈ A and x ∈ B}

For example, let A = { 2, 4, 6, 8 } and B = { 2, 3, 5, 8 }, then A ∩ B = { 2, 8 }



If A and B are two sets such that A ∩ B = φ, then A and B are called **disjoint sets**.

For example, let A = { 2, 4, 6, 8 } and B = { 1, 3, 5, 7 }. Then A and B are disjoint sets because there are no elements which are common to A and B.

Properties of intersection

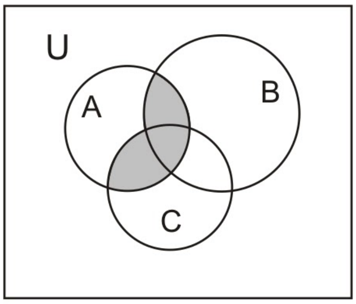
- $A \cap B = B \cap A$ (Commutative law).
- $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).
- $\mathrm{A} \cap \phi = \phi$,
- $\mathrm{A} \cap \mathrm{U} = \mathrm{A}$ (Law of ϕ and U).
- $\mathrm{A} \cap \mathrm{A} = \mathrm{A}$ (Idempotent law)
- If A is subset of B, then $A \cap B = A$

Distributive laws

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ i. e., ∩ distributes over ∪

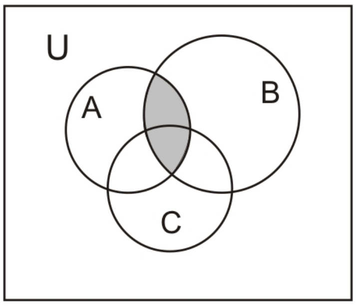
This can be seen easily from the following Venn diagrams

LHS:

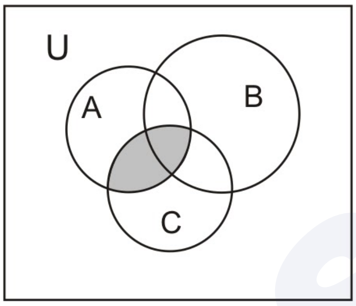


A ∩(BUC)

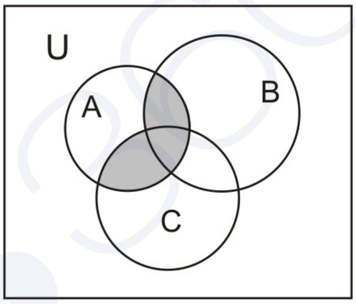
RHS:



(A∩B)



(A∩C)

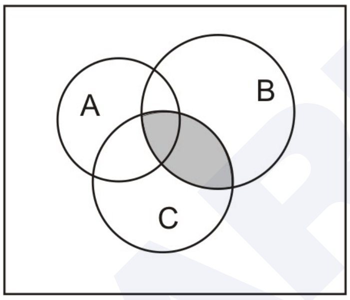


(A∩B)U(A∩C)

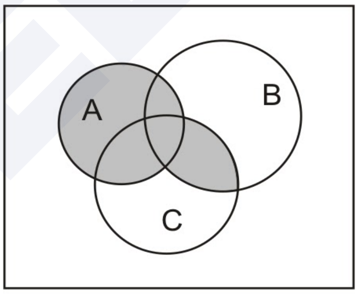
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ i. e., ∪ distributes over ∩

This can be seen easily from the following Venn diagrams

LHS:

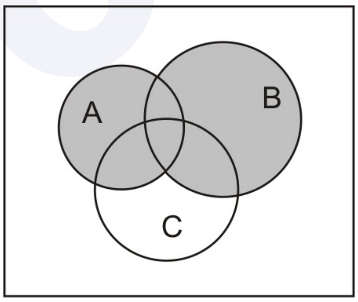


(B∩C)

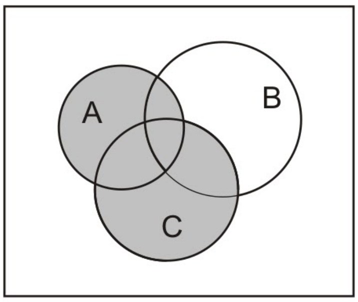


A U (B∩C)

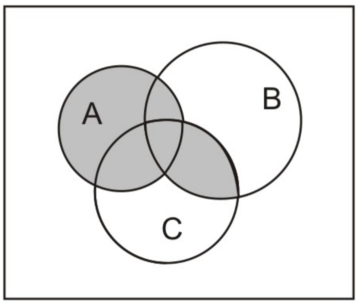
RHS:



(AUB)



(AUC)

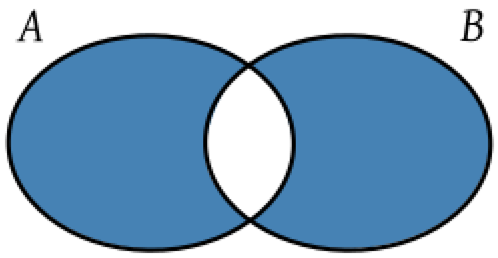


(AUB)∩(AUC)

Symmetric Difference of Sets ($A \Delta B$)

Symmetric difference of two sets A and B is defined as: $A \Delta B = (A - B) \cup (B - A)$

Venn Diagram



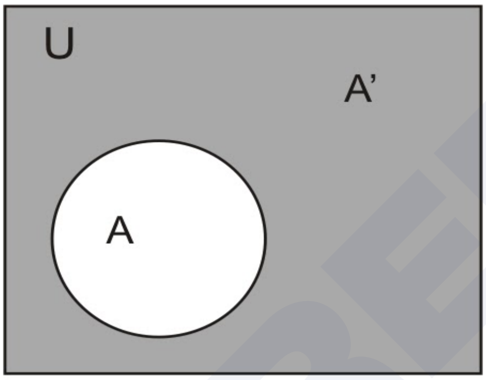
Clearly, A Δ B also equals (A ∪ B) - (A ∩ B)

Complement of a set

Let U be the universal set and A is a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A.

Symbolically, we use A' or A^C to denote the complement of A with respect to U.

A' = {x: x ∈ U and x ∉ A }. Obviously, A' = U – A



A' = Shaded One

Properties of Compliment

- A ∪ A′ = U
- A ∩ A′ = φ
- (A')' = A
- U′ = φ and φ′ = U
- A - B = A ∩ B'

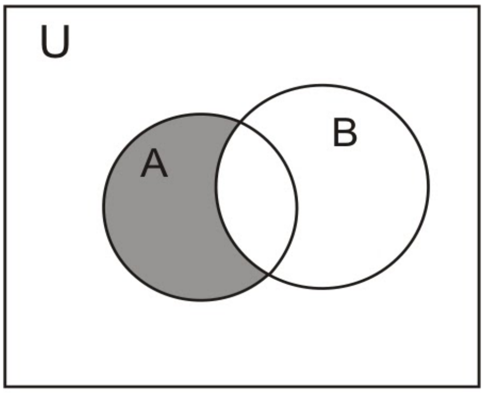
Difference of Sets

The difference of the sets A and B in this order is the set of elements which belong to A but not to B.

Symbolically, we write A – B and read as "A *minus* B".

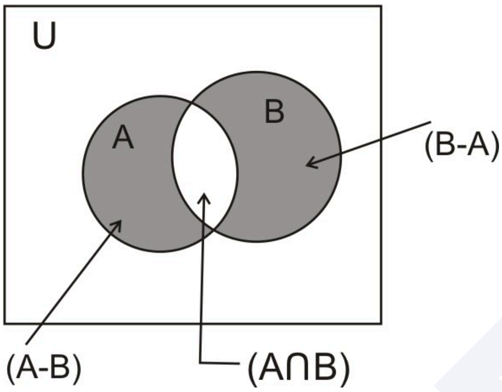
For example, If A = {1, 2, 3, 4} and B = {4, 5, 6, 8},

Then, A - B = {1, 2, 3} and B - A = {5, 6, 8}



(A-B)

The sets A – B, A ∩ B and B – A are mutually **disjoint sets**, i.e., the intersection of any two of these sets is the null set as shown in figure



Properties of Difference of Sets

1. In general A - B does not equal B - A

2. $\mathrm{A}-\mathrm{A}=\phi$

3. $\mathrm{A}-\phi=\mathrm{A}$

4. $\mathrm{A}-\mathrm{U}=\phi$

5. If A is a subset of B , then $\mathrm{A}-\mathrm{B}=\phi$

Cardinality

$n(A \cup B)=n(A)+n(B)–n(A \cap B)$

If $(A \cap B)=\varphi$, then $n(A \cup B)=n(A)+n(B)$

$n(A \cup B \cup C)=n(A)+n(B)+n(C)–n(A \cap B)–n(B \cap C)–n(A \cap C)+n(A \cap B \cap C)$

$n(A-B)=n(A)–n(A \cap B)$

De-Morgan's Laws

1. $(A \cup B)^{\prime}=A^{\prime} \cap B^{\prime}$

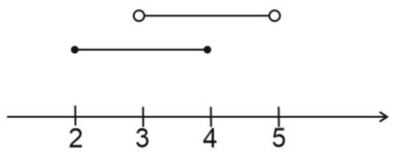
2. $(A \cap B)^{\prime}=A^{\prime} \cup B^{\prime}$

Union and intersection of Intervals

We can take union and intersection of intervals. Union of interval will contain all the numbers contained in all of the intervals involved.

eg. [2,4] ⋃ (3,5)

We draw number line, and represent these intervals at different heights



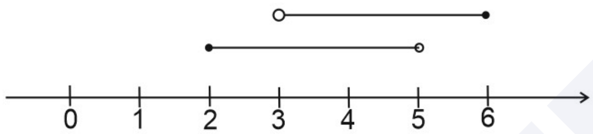
Now union will contain numbers in interval 1 as well as in interval 2

So, union is [2, 5)

For union of three intervals,

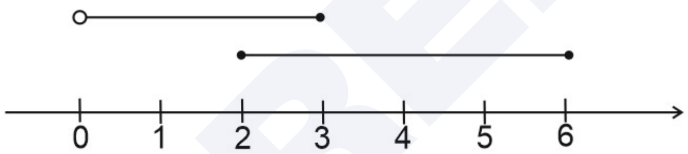
[2, 5) ⋃ (3,6] ⋃ (0, 3]

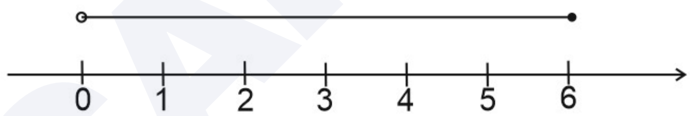
First we take union of first two intervals, [2,5) ⋃(3,6]



So we get [2, 6]

Now we can take union of this with third interval, [2, 6] ⋃ (0, 3]



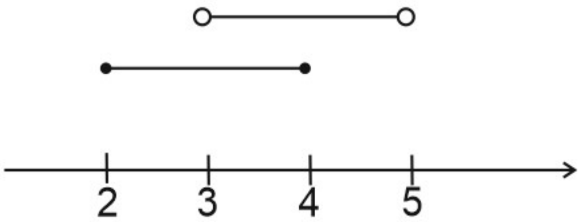


So, union is (0,6]

Intersection of intervals

In intersection we keep only the number common in the intervals involved

eg. [2,4] ⋂ (3,5)



Intersection (common part) is (3,4]   (Note: 3 is not included since 3 is not lying in second interval)

For three intervals, first find intersection of any two intervals and then find intersection of resultant with third interval.

Chapter 2: Relations & Functions

Ordered pair

A pair of elements grouped together in a particular order is known as an ordered pair.

e.g. : (a,b), (3,5), (-1,0)...

The ordered pair (a, b) and (b, a) are different.

Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

i.e. (x, y) = (u, v) if and only if x = u, y = v.

Cartesian product

The cartesian product of two non-empty sets A and B is the set of all ordered pairs (x, y), where x ∈ A and y ∈ B.

Symbolically, we write it as A × B and it is read as ‘A cross B’.

A × B = {(a,b): a ∈ A, b ∈ B}

For example, If $A=\{1,2\}$ and $B=\{a,b\}$

Then $A\times B=\{(1,a),(1,b),(2,a),(2,b)\}$

Note

- A × A × A = {(a, b, c) : a, b, c ∈ A}. Here (a, b, c) is called an ordered triplet.
- R × R = {(x,y): x,y ∈ R} and R × R × R = (x,y,z): x,y,z ∈ R}

Number of elements in A x B

If there are p elements in A and q elements in B, then there will be pq elements in $A\times B$,

i.e., if $n(A)=p$ and $n(B)=q$, then $n(A\times B)=pq$

Number of Relations from A to B

If A have m elements and B have n elements, then $A\times B$ has $m\times n$ element.

As the number of subsets of $A \times B$ is 2^{mn}, and a relation is a subset of $A \times B$, so the total number of relations from A to B will be 2^{mn}.

Relation on a set

If a relation is from A to A itself, then this relation is called **relation on set A.**

Empty Relation

A relation R **on a set A** is called empty relation, if no element of A is related to any element of A, i.e., R = φ

For example, Let A = {2, 4, 6} and R = {(a, b) : a, b ∈ A and a + b is odd}

Here, R contains no element, therefore R is an empty relation on A

Universal Relation

A relation R **on a set A** is called universal relation, if each element of A is related to every element of A, i.e., R has all the ordered pair contained in A x A

So, R = A × A.

For example,

1. Let A = {2 ,4} and R = {(2,2), (2,4), (4,2), (4,4)}

Here, R = A x A. Hence, R is Universal relation

2. Let A = {1,2,3}, and R = {(a, b) : |a - b| > -2, a , b ∈ A}

Clearly mod value of difference of any pair (a,b) will be greater than -2

So, each possible ordered pair in A x A will lie in R, therefore R is a universal relation

Identity relation

If every element of A is related to itself **only,** then it is known as an identity relation on A . It is denoted by I_A

R = {(a, b) : a ∈ A, b ∈ A and a = b}

It can also be written as $I_A = \{(a, a) \ : \ a \in A\}$

For example A = {2 ,4, 6}

Then, I_A = {(2,2), (4,4), (6,6)}

Reflexive Relation

- A relation R on a set a is **reflexive** if every element is related to itself.
- For all $a \in A$,
  $(a, a) \in R$

Symmetric Relation

- A relation R on a set a is **symmetric** if whenever an element a is related to b, then b is related to a.
- For all $a, b \in A$,
  $(a, b) \in R \implies (b, a) \in R$

Transitive Relation

- A relation R on a set a is **transitive** if whenever an element a is related to b, and b is related to c, then a is related to c.
- For all $a, b, c \in A$,
  $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$

Equivalence Relation

For all $a, b, c \in A$,

$$\begin{cases} (a, a) \in R & \text{(Reflexive)} \\ (a, b) \in R \implies (b, a) \in R & \text{(Symmetric)} \\ (a, b) \in R \text{ and } (b, c) \in R \implies (a, c) \in R & \text{(Transitive)} \end{cases}$$

Function: A function f from set A to set B assigns each element in A exactly one element in B.

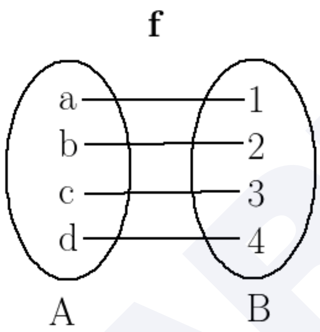
$f : A \rightarrow B$

A relation from a set A to a set B is said to be a function from A to B if every element of set A has one and only one image in set B.

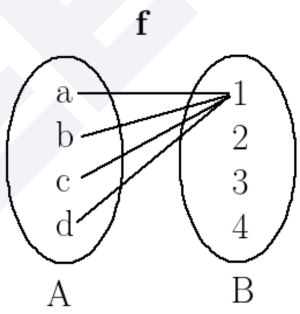
OR

A and B are two non-empty sets, then a relation from A to B is said to be a function if each element x in A is assigned a unique element f(x) in B, and it is written as

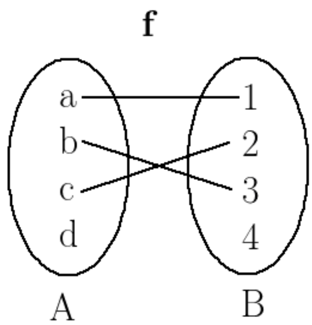
f : A → B and read as f is mapping from A to B.



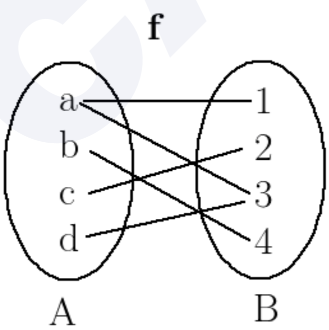
Function



Function



Not a function



Not a function

Third one is not a function because, d is not related(mapped) to any element in B.

Fourth is not a function as element a in A is mapped to more than one element in B.

If f is a function from A to B and (a, b) belongs to f, then f (a) = b, where 'b' is called the image of 'a' under f and 'a' is called the pre-image of 'b' under f.

In the ordered pair (1,2). 1 is the pre-image of 2.
Number of function from A to B

Let set $A=\{x_1, x_2, x_3 \ldots \ldots \ldots \ldots \ldots, x_m\}$ i.e. m elements and $B=\{y_1, y_2, y_3 \ldots \ldots \ldots \ldots \ldots \ldots y_n\}$ n elements

Total number of function from A to B$= n^m$

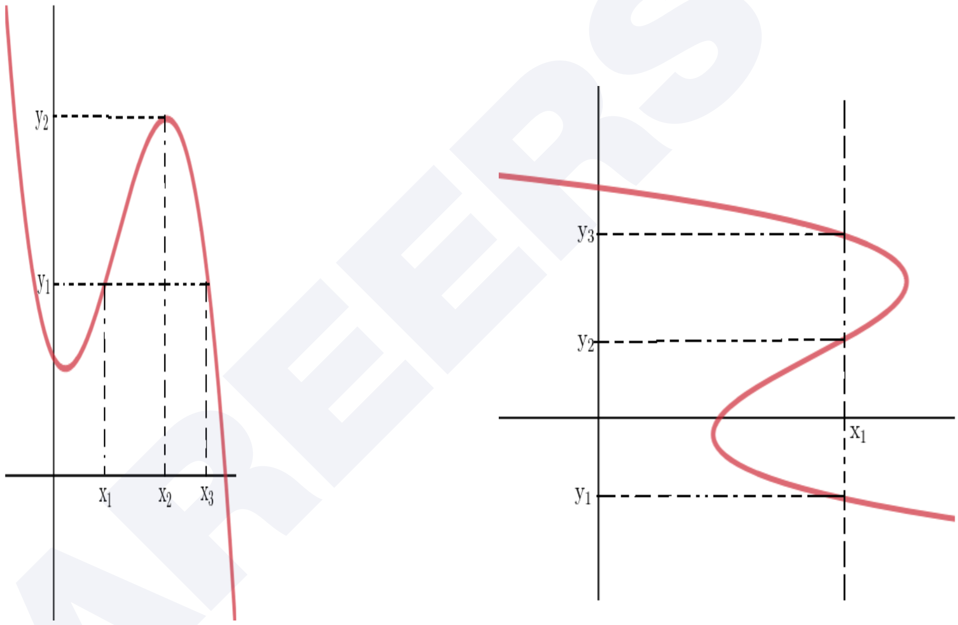
(The proof of this formula requires the use of Permutation and Combination, so it will be covered later)

Vertical Line Test

Functionality check using the graph:

If any line drawn parallel to y-axis cuts the curve at most one point, then it is a function.

If any such line cuts the graph at more than one point, then it is not a function.



In figure 1, any line parallel to y-axis cuts the curve at one point only. Each value of x would have one and only one image (value of y), so figure 1 is a function.

Whereas in figure 2, a line parallel to y-axis cuts the curve in three points. Here for x = x_1 , we have three images i.e. y_1, y_2, and y_3 . Therefore, figure 2 is not a function.

Types of function

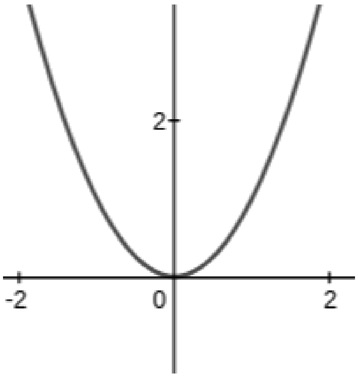
Algebraic function:

A function f is said to be algebraic if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots. e.g. $f(x)=\sqrt{1+x}$

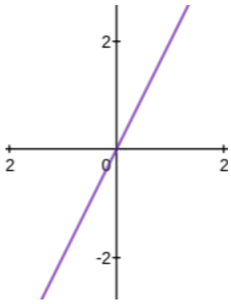
Monomial function

A function of the form $y = ax^n$, where **a** is constant and **n** is a non-negative integer, is called a monomial function.

e.g $y = x^2, y = 2x, y = -x$, etc



$y = x^2$



$y = 2x$

Polynomial function

A real valued function f : R → R defined by $y = f(x) = a_0 + a_1x + a_2x^2 ... + a_nx^n$, where n ∈ N, and a_0 , a_1 , a_2 ...a_n ∈ R, for each x ∈ R, is called Polynomial functions.

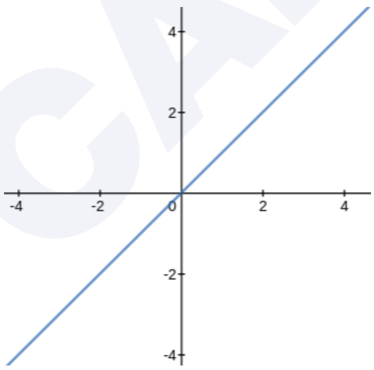
The higest power of x is called the Degree of this polynomial.

- Domain for such functions is R.
- The range depends on the degree of the polynomial. If degree is odd, then range is R, but it is does not equal R if the degree is even.

Identity function

Let R be the set of real numbers. Define the real valued function f : R → R by y = f(x) = x for each x ∈ R.

Such a function is called the identity function. It is denoted by I_A. Here the domain and range of function are R. The graph is a straight line.



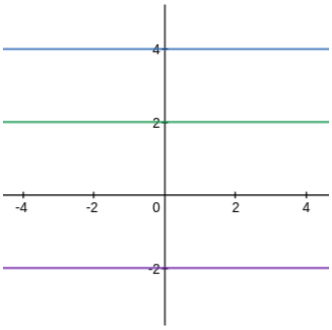
y = x

Constant function

The function f: R → R by y = f (x) = c, x ∈ R where c is a constant and each x ∈ R.

Here domain of f is R and its range is {c}.

The graph is a line parallel to the x-axis. For example, if f(x) = 4 for each x ∈ R, then its graph will be a line as shown in Fig



As from the above figure, we can see that blue line is y = 4

Green line is y = 2 and purple line is y = -2

Rational Function

$f(x)=\frac{\rho(x)}{q(x)}$, Where $\rho(x)$ and $q(x)$ polynomials in x .

- Domain of this function is $R-\{x: q(x)=0\}$
- Range depends on the function.

Modulus Function:

The function f: R→R defined by f(x) = |x| for each $x \in R$ is called modulus function.

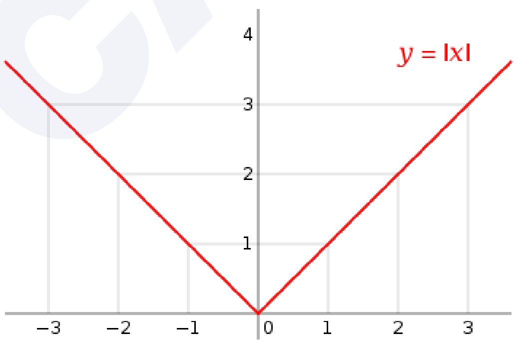
For each non-negative value of x, f(x) is equal to x. But for negative values of x, the value of f(x) is the negative of the value of x

$|\mathrm{x}|, \mathrm{x} \in \mathbb{R}=\left\{\begin{array}{cc}

x, & x \geq 0 \

-x, & x<0

\end{array}\right$



Range $\in[0, \infty)$

Modulus Equations: Properties

If a > 0

1. |x| = a, then x = a, -a

2. |x| = |-x|

3. $|x|^2 = x^2$

4. If |x| = x, then x > 0 or x=0

5. If |x| = -x, then x < 0 or x=0

6. |f(x)| = |g(x)| , then f(x) = g(x) or f(x) = - g(x)

Signum function:

The function $f : R \to R$ defined by

$\operatorname{sgn}(\mathrm{x})=\left\{\begin{array}{ccl}

1 & \text { if } & x>0 \

-1 & \text { if } & x<0 \

0 & \text { if } & x=0

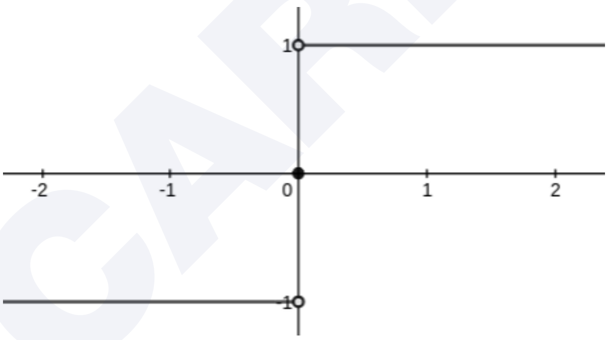
\end{array}\right.$

is called the signum function. The domain of the signum function is R and the range is the set {-1,0,1}.

This function can also be written in another form:

$$\operatorname{sgn}(x)=\left\{\begin{array}{c}\frac{|x|}{x}, x \neq 0 \\ 0, x=0\end{array}\right\}$$

Graph:



Range $\in \{-1,0,1\}$

Greatest integer function (G.I.F.)

The function f: R $\to$ R defined by f(x) = [x], x $\in$ R assumes the value of the greatest integer which is equal to or less than x. Such a functions is called the greatest integer function.

eg;

[1.75] = 1

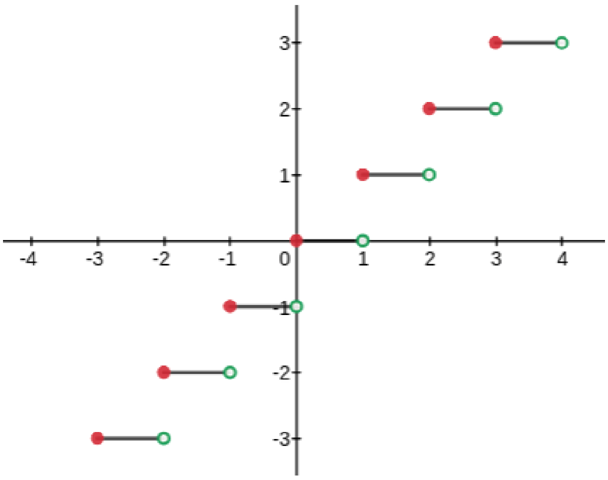
[2.34] = 2

[-0.9] = -1

[-4.8] = -5

[4] = 4

[-1] = -1
Graph:



From the definition of [x], we

can see that

[x] = –1 for –1 ≤ x < 0

[x] = 0 for 0 ≤ x < 1

[x] = 1 for 1 ≤ x < 2

[x] = 2 for 2 ≤ x < 3 and so on.

Properties of greatest integer function:

i) [a] = a (If a is an integer)

ii) [[x]] = [x]

iii) x-1 < [x] ≤ x

iv) [x + a] = [x] + a (If a is an integer)

v) [x - a] = [x] - a (If a is an integer)

vi) $[x]+[-x]=\left\{\begin{array}{ccc}

0, & \text { if } & x \in Z \

-1, & \text { if } x \notin Z &

\end{array}\right.$

Fractional part function:

$\{x\} = x - [x]$

When [x] is the Greatest Integer Function e.g.

{2.2} = 2.2 - [2.2] = 2.2 - 2 = 0.2

{1.7} = 1.7 - [1.7] = 1.7 - 1 = 0.7

{2} = 2 - [2] = 2 - 2 = 0

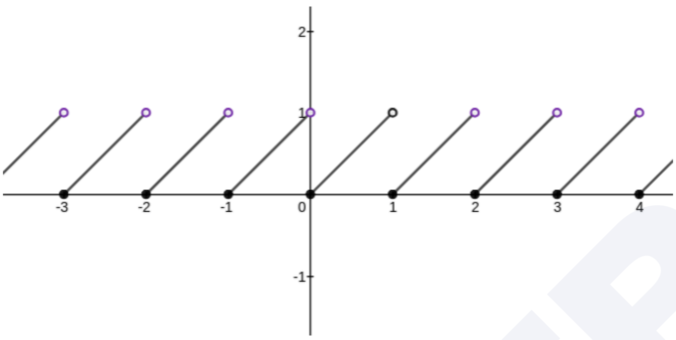
{ - 2.2} = -2.2 - [-2.2] = 2.2 - (-3) = 0.8

{ - 1.7} = -1.7 - [- 1.7] = 1.7 - (-2) = 0.3

{ - 2} = - 2 - [-2] = 2 - (- 2) = 0

Clearly , 0 ≤ {x} < 1

Graph



Domain: R

Range $\in [0,1)$

Properties of fractional part of x

i) {x} = x if 0 ≤ x < 1

ii) {a} = 0, if a is an integer

iii) 0 ≤ {x} < 1

iv) { x + a } = {x}   (If a is an integer)

v) {x} + {-x} = 1, if x doesn't belongs to integer

vi) {x} + {-x} = 0, if x belongs to integer

Even function: If for a function f(x),

$f(-x)=f(x)$, then the function is known as even function. Even functions are symmetric about the y axis.

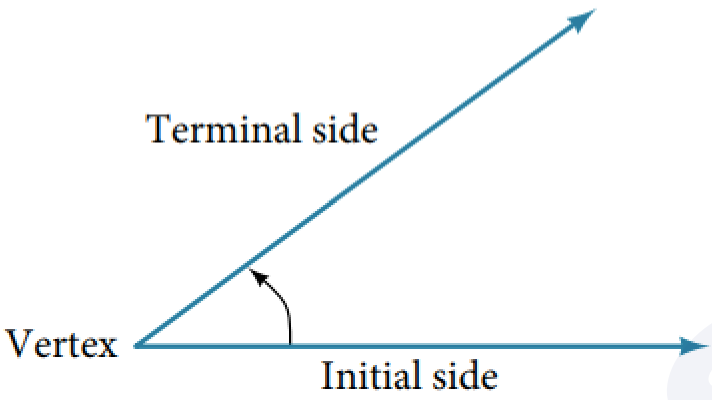
Odd function: If for a function f(x),

$f(-x)=-f(x)$ then the function is known as odd function. Odd functions are symmetric about the origin.

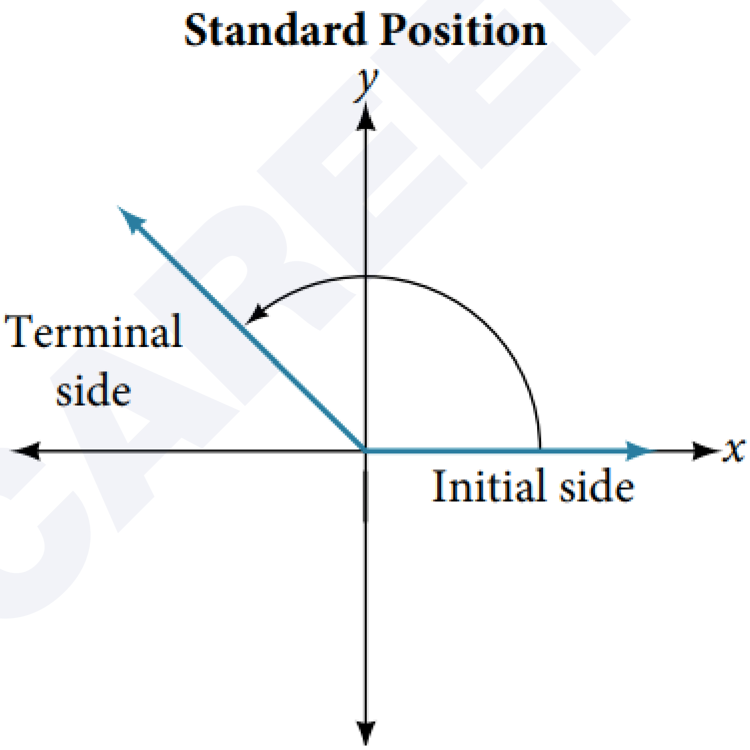
Chapter 3: Trigonometry

Measurement of Angle

To form an angle, we start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the initial side, and the rotated ray is the terminal side. And, the measure of an angle is the amount of rotation from the initial side to the terminal side.



An angle in standard position if its vertex is located at the origin, and its initial side extends along the positive x-axis, as you can see from the figure given below.



If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a positive angle. If the angle is measured in a clockwise direction, the angle is said to be a negative angle.

There are three systems used for the measurement of angles

1. Sexagesimal system

In this system, an angle is measured in degrees, minutes and seconds.

1 Right angle = 90^{o} (Read as 90 degree)

1^{o} = 60’ (1 degree = 60 minutes)

1’ = 60” (1 minutes = 60 seconds)

2. Centesimal system

In this system, the measurement of the right angle is divided into 100 equal parts, and parts called Grades.

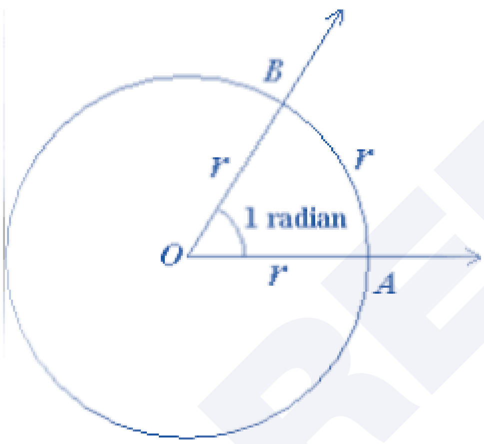
1 Right angle = 100g (Read as 100 grades)

3. Circular system

In this system, an angle is measured in radians.

One radian is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii.

The formula for radian measure of an angle formed by an arc of length l at the centre of circle of radius r is (Length of arc)/(Radius) = l/r



Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians.

So, 2π radians = 360°

So, π radians = 360°/ 2 = 180°

and 1 radian = 180°/ π ≈ 57.3°

Interconversion of units

Since, degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion (where θ is the measure of the angle in degrees and θ_R is the measure of the angle in radians)

$$\frac{\theta}{180} = \frac{\theta_{\mathrm{R}}}{\pi}$$

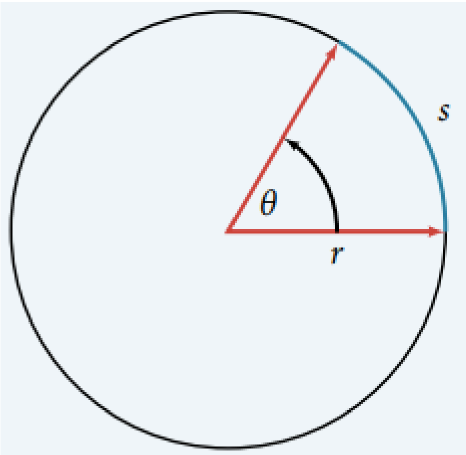
OR

$$\frac{\text{Degree}}{180} = \frac{\text{Radians}}{\pi}$$

Note:

1. Radian is the unit to measure angles, and it does not mean that π stands for 180^o. π is a real number. Remember the relation, π radians = 180^o.
2. In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians. Arc length = (radius) x (Angle subtended by an arc in radians)

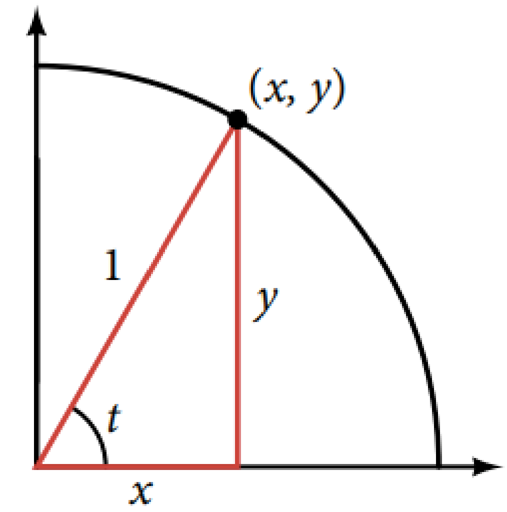
s = rΘ.

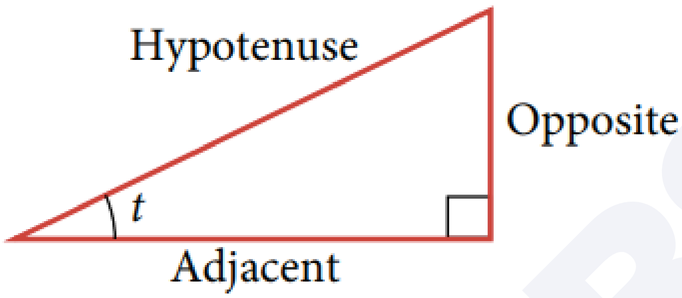


Trigonometric Ratios

Trigonometric Functions of Acute Angles

We can define the trigonometric functions in terms of an angle t and the lengths of the sides of the triangle. The adjacent side (=x) is the side closest to the angle (Adjacent means “next to.”). The opposite side (=y) is the side across from the angle. The hypotenuse (=1) is the side of the triangle opposite the right angle.





Sine $\sin t=\frac{\text{opposite}}{\text{hypotenuse}}$

Cosine $\cos t=\frac{\text{adjacent}}{\text{hypotenuse}}$

Tangent $\tan t=\frac{\text{opposite}}{\text{adjacent}}$

Reciprocal Functions: In addition to sine, cosine, and tangent, there are three more functions. These too are defined in terms of the sides of the triangle.

Cosecant $\csc \mathrm{t}=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{1}{\sin \mathrm{t}}$

Secent $\sec \mathrm{t}=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{1}{\cos \mathrm{t}}$

Cotangent $\cot t=\frac{\text{adjacent}}{\text{opposite}}=\frac{1}{\tan t}$

Since, the hypotenuse is the greatest side in a right angle triangle, $\sin t$ and $\cos t$ can never be greater than unity and $\operatorname{cosec} t$ and $\sec t$ can never by less than unity.

Trigonometric Ratios of some Special Angles

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Trigonometric Identities

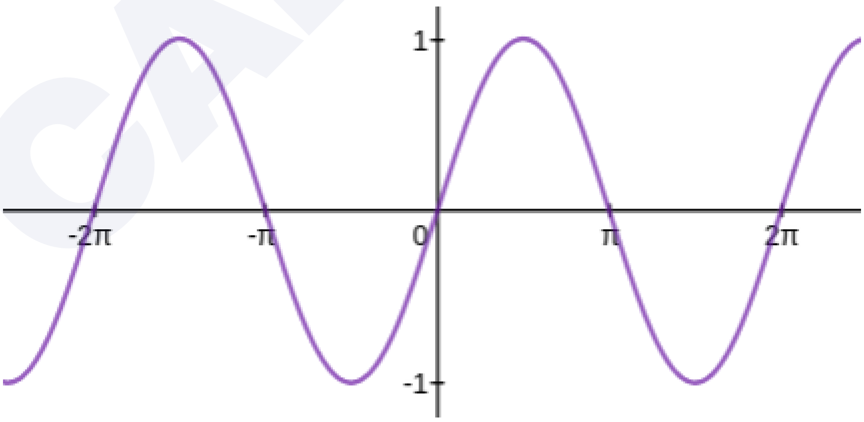
These identities are the equations that hold true regardless of the angle being chosen.

$$\sin^2 t + \cos^2 t = 1$$
$$1 + \tan^2 t = \sec^2 t$$
$$1 + \cot^2 t = \csc^2 t$$
$$\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t}$$

Graph of Trigonometric Function

Sine Function

y = f(x) = sin(x)

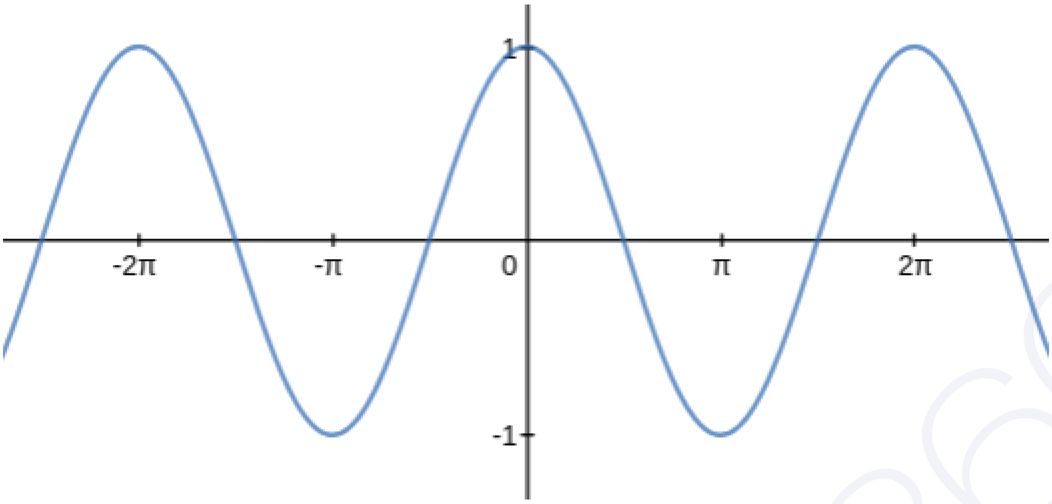


Domain is R

Range is [-1, 1]

Cosine Function

y = f(x)= cos(x)

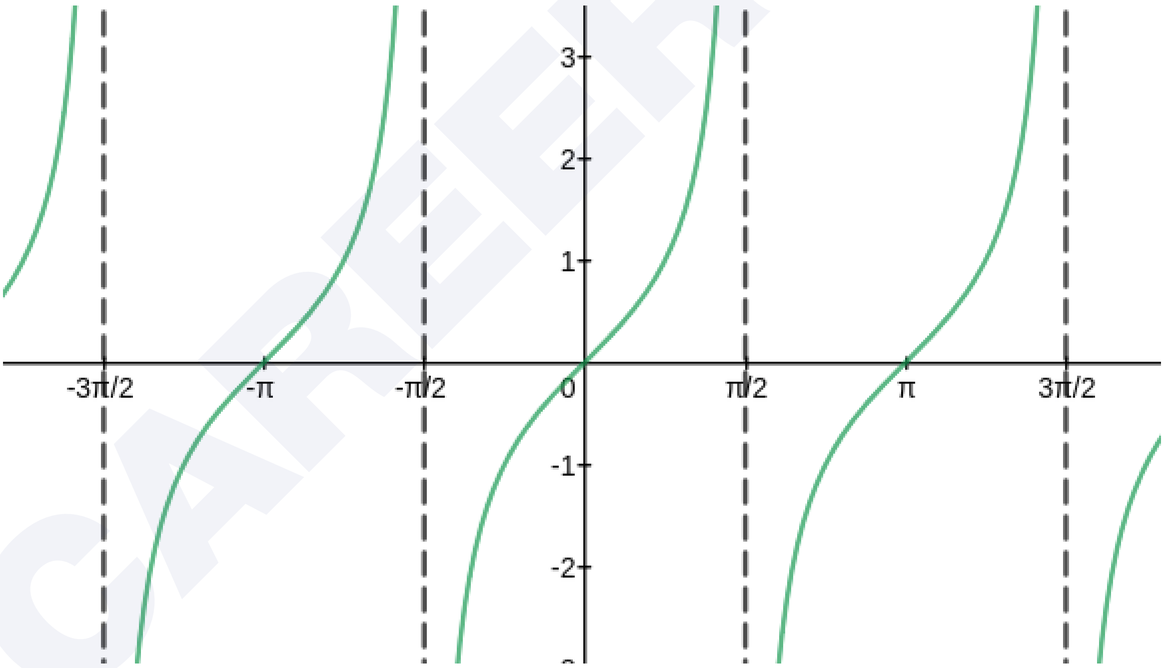


Domain is R

Range is [-1, 1]

Tangent Function

y = f(x) = tan(x)

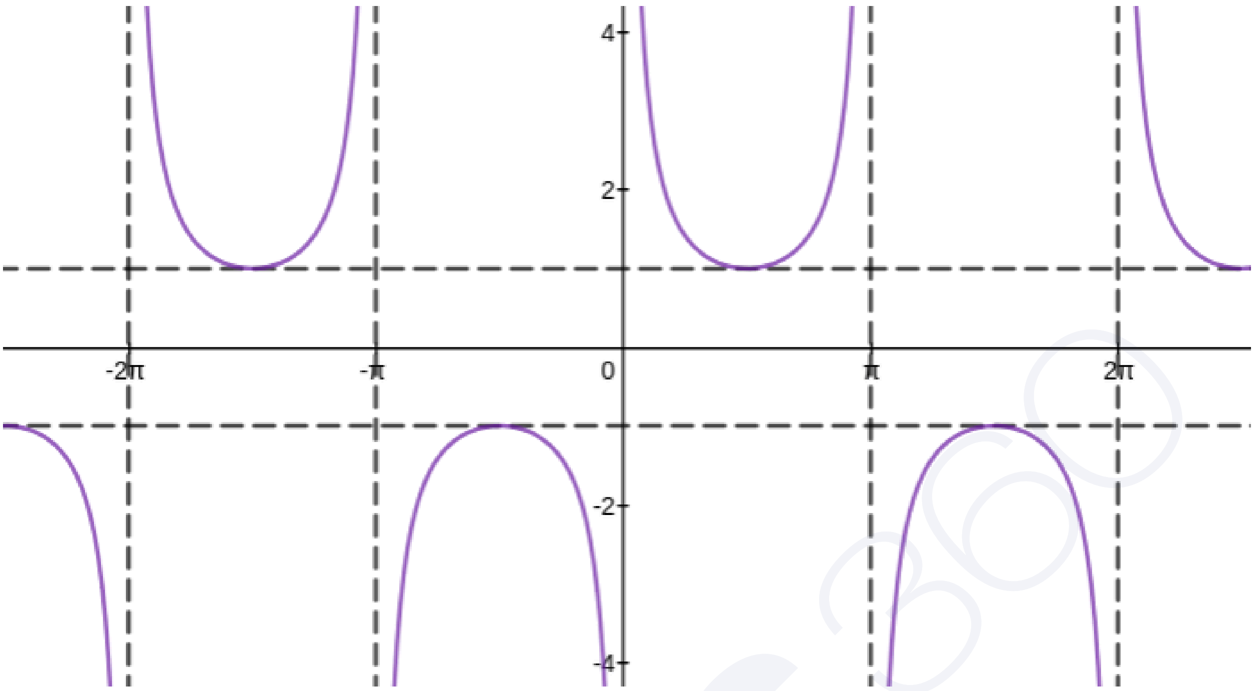


$$\text{Domain is } \mathbb{R} - \left\{ \frac{(2n+1)\pi}{2},\ n \in \mathbb{I} \right\}$$

Range is R

Cosecant Function

y = f(x) = cosec(x)



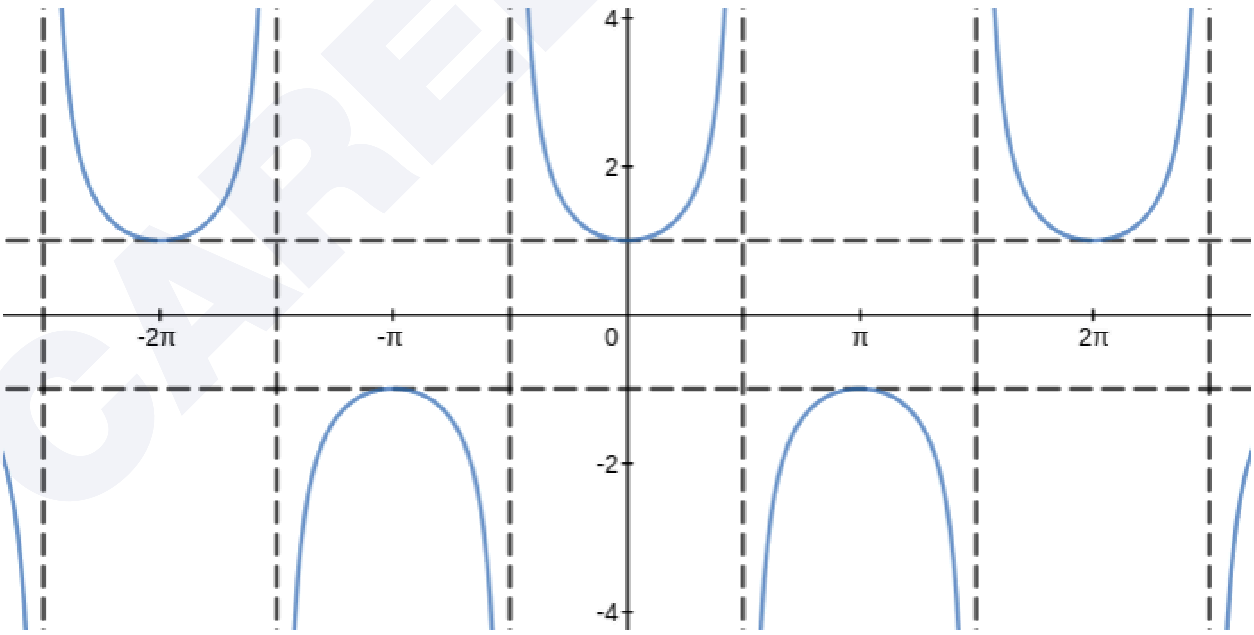
Domain is R - {nπ, n ∈ I (Integers)}

Range is R - (-1, 1)

Period is 2π

Secant Function

y = f(x) = sec(x)



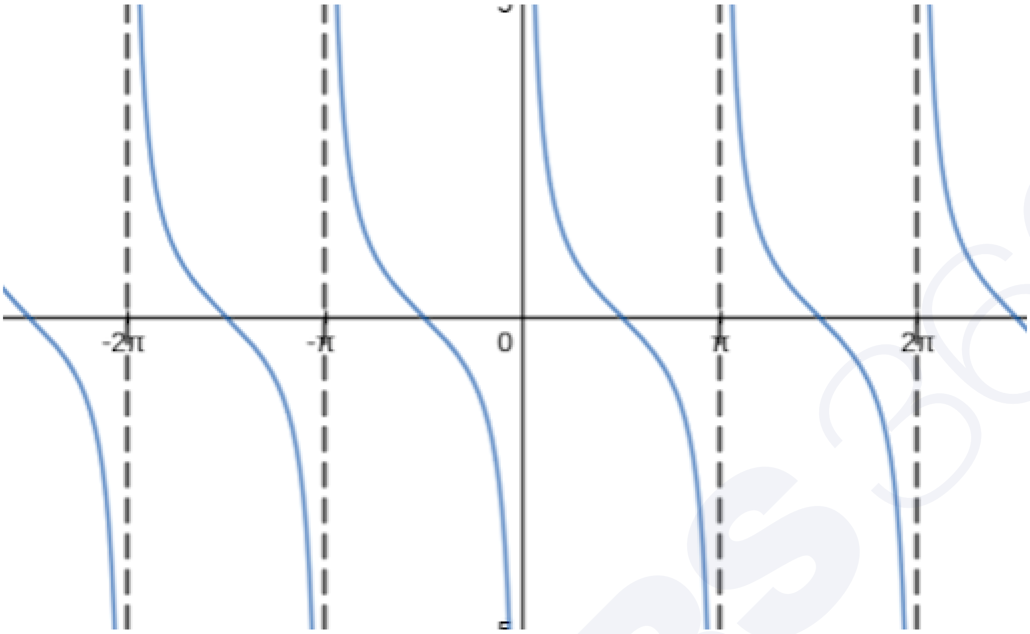
$$\text{Domain is } \mathbb{R} - \left\{\frac{(2n+1)\pi}{2},\ n \in \mathbb{I}\right\}$$

The range is R - (-1, 1)

Period is 2π

Cotangent Function

y = f(x) = cot(x)



Domain is R - {nπ, n ∈ I (Integers)}

Range is R

Period is 2π

Allied Angles:

$\sin(180^\circ - \theta) = \sin\theta, \quad \cos(180^\circ - \theta) = -\cos\theta$

$\tan(180^\circ - \theta) = -\tan\theta$

(similar rules apply for $90^\circ \pm \theta$, $270^\circ \pm \theta$, etc.)

Trigonometric Ratios of Compound Angles:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Sum to Product:

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Product to Sum:

$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

Double Angle Formula:

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Triple Angle Formula:

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

Reduction Formula:

$\sin(n\pi \pm \theta) = \pm \sin\theta, \quad \cos(n\pi \pm \theta) = \pm \cos\theta$

Half Angle Formula:

$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}, \quad \cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$

Trigonometric Series (Example):

$\sin A + \sin(A + d) + \cdots + \sin[A + (n-1)d] = \frac{\sin\left(\frac{nd}{2}\right)\cdot\sin\left[A+\frac{(n-1)d}{2}\right]}{\sin\left(\frac{d}{2}\right)}$

Conditional Identities:

$\sin A + \sin B + \sin C = 4\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)$ (for $A + B + C = 0$)

Maximum and Minimum Values:

For $f(\theta) = a\sin\theta + b\cos\theta$, max = $\sqrt{a^2 + b^2}$, min = $-\sqrt{a^2 + b^2}$

General Solution of Standard Equations:

$\sin x = 0 \Rightarrow x = n\pi$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2} + n\pi$

$\tan x = 0 \Rightarrow x = n\pi$

(where $n \in \mathbb{Z}$)

Heights and Distances:

Use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ or $\sin, \cos$ rules in right triangle setups

Inverse Trigonometric Identities:

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$$

Complementary Angles:

$$\sin(90^\circ - \theta) = \cos\theta, \quad \tan(90^\circ - \theta) = \cot\theta$$

$$\sec(90^\circ - \theta) = \csc\theta$$

Chapter 4: Complex Numbers & Quadratic Equations

Complex Numbers

A number of the form a + ib is called a complex number (where a and b are real numbers and i is iota). We usually denote a complex number by letter z, z_1, z_2, etc

For example, z = 5 + 2i is a complex number.

5 here is called the real part and is denoted by Re(z), and 2 is called imaginary part and is denoted by Im(z)

Note: 2i is not the imaginary part, only 2 is called the imaginary part.

For z = - 2 - i, Re(z) = -2, Im(z) = -1

We denote the set of all complex numbers by C.

Purely Real and Purely Imaginary Complex Number

- A complex number is said to be purely real if its imaginary part is zero, Im(z) = 0

  i.e. z = 4 + 0i, z = 4.

- A complex number is said to be purely imaginary if its real part is zero, Re(z) = 0

  i.e. z = 0 + 3i, z = 3i

All real numbers are also complex numbers (with b=0). e.g. 4 can be written as 4 + 0i

So, R is a proper subset of C.

Equality of Complex Numbers

Two complex numbers are said to be equal if and only if their real parts are equal and their imaginary parts are equal.

a + ib = c + id

⇒ a = c and b = d

a, b, c, d ∈ R and i = √-1

Note: In complex number, inequalities do not exist. $z_1 > z_2$ does not make any sense in complex numbers

e.g.,

- 4 + 3i > 1 + i is wrong (as two complex numbers cannot be compared)
- 1000 + 7000i > -39 - 800i is also wrong
- 1000 + 40i > 2 is wrong

We can only compare two complex numbers if their imaginary parts are 0. e.g., 5 + 0i > 4 + 0i is correct.

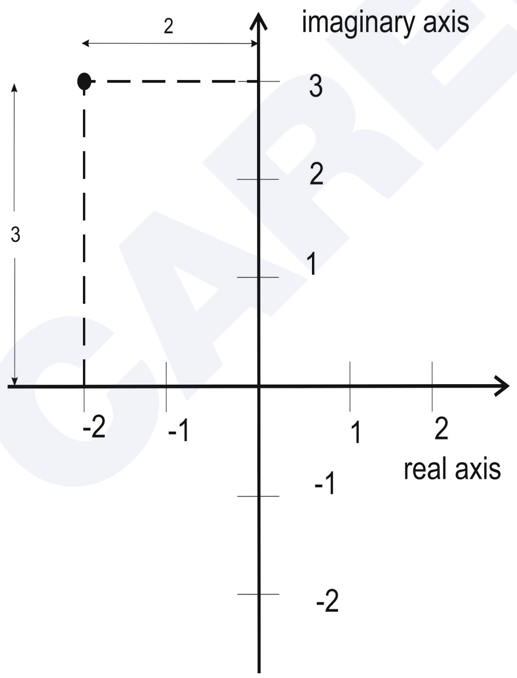
Argand Plane

A complex Number can be represented on a rectangular coordinate system called Argand Plane.

In this z = a + ib is represented by a point whose coordinates are (a,b)

So, x-coordinate of the point is the Real part of z and y coordinate is imaginary part of z

e.g. z = - 2 + 3i is represented by the point (-2,3) and it lies in second quadrant.



Integral Powers of Iota:

For $n \in \mathbb{Z}$,

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i$$

Complex Numbers as Ordered Pairs:

A complex number can be represented as an ordered pair of real numbers (x, y), where

$z = x + iy \quad \text{with } x, y \in \mathbb{R} \text{ and } i^2 = -1$

Conjugate

The conjugate of a complex number z = a + ib (a, b are real numbers) is a − ib. It is denoted as $\bar{z}$.

i.e. if z = a + ib, then its conjugate is $\bar{z}$ = a - ib.

Conjugate of complex numbers is obtained by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

Note:

- When a complex number is added to its complex conjugate, the result is a real number. i.e. z = a + ib, $\bar{z}$ = a - ib

  Then the sum, z + $\bar{z}$= a + ib + a - ib = 2a (which is real)

- When a complex number is multiplied by its complex conjugate, the result is a real number i.e. z = a + ib, $\bar{z}$ = a - ib

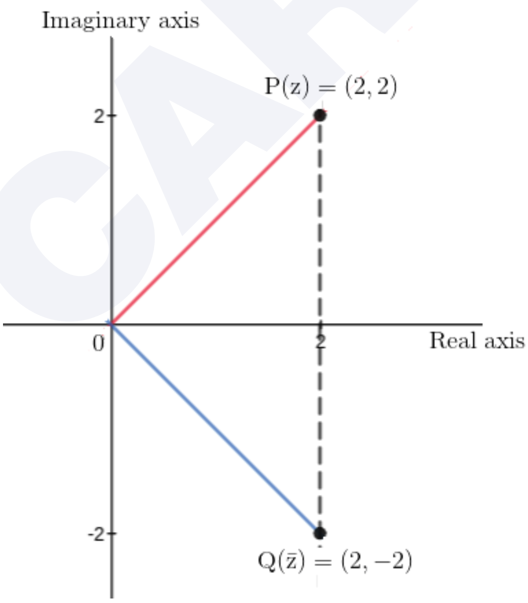
  Then the product, z • $\bar{z}$ = (a + ib) • (a - ib) = a^2 - $(ib)^2$

  = a^2 + b^2 (which is real)

Geometrically complex conjugate of a complex number is its mirror image with respect to the real axis (x-axis).

For example

z = 2 + 2i and $\bar{z}$ = 2 - 2i



Properties of the conjugate complex numbers:

z, z_1, z_2, and z_3 be the complex numbers

1. $\overline{(\bar{z})} = z$

2. $z + \bar{z} = 2 \cdot \mathrm{Re}(z)$

3. $z - \bar{z} = 2i \cdot \mathrm{Im}(z)$

4. $\mathrm{z} + \overline{\mathrm{z}} = 0 \Rightarrow \mathrm{z} = -\overline{\mathrm{z}} \Rightarrow \mathrm{z}$ is purely imaginary

5. $\mathrm{z} - \overline{\mathrm{z}} = 0 \Rightarrow \mathrm{z} = \overline{\mathrm{z}} \Rightarrow \mathrm{z}$ is purely real

6. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

In general, $\overline{z_1 \pm z_2 \pm z_3 \pm \ldots\ldots\ldots\ldots \pm z_n} = \overline{z_1} \pm \overline{z_2} \pm \overline{z_3} \pm \ldots\ldots\ldots\ldots \pm \overline{z_n}$

7. $\overline{\mathrm{Z}_1 \cdot \mathrm{Z}_2} = \overline{\mathrm{Z}_1} \cdot \overline{\mathrm{Z}_2}$

In general, $\overline{\mathrm{Z}_1 \cdot \mathrm{Z}_2 \cdot \mathrm{Z}_3 \cdot \ldots\ldots\ldots\ldots \cdot \mathrm{Z}_n} = \overline{\mathrm{Z}_1} \cdot \overline{\mathrm{Z}_2} \cdot \overline{\mathrm{Z}_3} \cdot \ldots\ldots\ldots\ldots \cdot \overline{\mathrm{Z}_n}$

8. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \quad z_2 \neq 0$

9. $\overline{\mathrm{z}^{\mathrm{n}}} = (\overline{\mathrm{z}})^{\mathrm{n}}$

10. $\mathrm{z}_1 \cdot \overline{\mathrm{z}_2} + \overline{\mathrm{z}_1} \cdot \mathrm{z}_2 = 2\,\mathrm{Re}\left(\mathrm{z}_1 \cdot \overline{\mathrm{z}_2}\right) = 2\,\mathrm{Re}\left(\overline{\mathrm{z}_1} \cdot \mathrm{z}_2\right)$

Modulus of a Complex Number:

The modulus (or absolute value) of z is the distance from the origin to the point (x, y):

$|z| = \sqrt{x^2 + y^2}$

Argument of a Complex Number:

The argument (or amplitude) θ of z is the angle made with the positive real axis:

$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad x \neq 0$

Algebra of Complex Numbers:

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Closure Law: Product of any two complex numbers is again a complex number.

$z_1, z_2 \in \mathbb{C} \Rightarrow z_1 \cdot z_2 \in \mathbb{C}$

Commutative Law: $z_1 \cdot z_2 = z_2 \cdot z_1$

Associative Law: $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$

Multiplicative Identity: 1 is the identity element for multiplication in $\mathbb{C}$.

$z \cdot 1 = z = 1 \cdot z$

Multiplicative Inverse: For $z = a + ib \neq 0$, the inverse is:

$z^{-1} = \frac{1}{z} = \frac{a-ib}{a^2+b^2}$

Distributive Law: $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$

Conjugate of a Complex Number:

$\bar{z} = a - ib \quad \text{if } z = a + ib$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0$

$z \cdot \bar{z} = |z|^2 = a^2 + b^2$

$\bar{\bar{z}} = z$

Polar Form of a Complex Number:
If $z = a + ib$, then
$z = r(\cos\theta + i\sin\theta)$
$r = |z| = \sqrt{a^2 + b^2}, \quad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

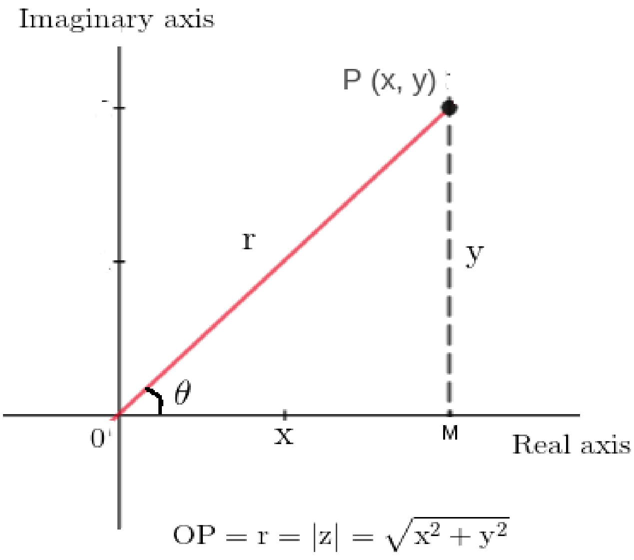
In polar form, we represent the complex number through the argument and modulus value of complex numbers.

Let z = x + iy be a complex number,

And we know that

$|z| = \sqrt{x^2 + y^2} = r$

And let arg(z) = θ



From the figure, x = |z| cos(θ) = r cos(θ)

and y = |z| sin(θ) = r sin(θ)

So, z = x + iy =   r cos(θ) + i.r sin(θ) = r (cos(θ) + i.sin(θ))

This form is called polar form with $\theta=$ principal value of arg(z) and r = |z|.

For general values of the argument

$\mathrm{z}=\mathrm{r}[\cos (2 \mathrm{n} \pi+\theta)+i \sin (2 \mathrm{n} \pi+\theta)]$, where $n \in$ Integer

Euler's Form of a Complex Number:
$z=r e^{i \theta}$
where $r=|z|$ and $\theta=\arg (z)$

The polar form of complex number $z=r(\cos \theta+i \sin \theta)$

In Euler form $(\cos \theta+i \sin \theta)$ part of the polar form of complex numbers is represented by $e^{i\Theta}$. So, $z=r(\cos \theta+i \sin \theta)$ is written as r.$e^{i\Theta}$ in Euler's Form

We know the expansion of e^x is

The expansion of e^x is

$

e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots .

$

Replacing x by ix

$

\begin{aligned}

& e^{i x}=1+\frac{i x}{1!}+\frac{(i x)^2}{2!}+\frac{(i x)^3}{3!}+\frac{(i x)^4}{4!}+\ldots \

& e^{i x}=1+\frac{i x}{1!}-\frac{x^2}{2!}-\frac{i x^3}{3!}+\frac{x^4}{4!}+\ldots+

\end{aligned}

$

rearranging the terms, we have

$

e^{i x}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)

$

We notice that first bracket is the expansion of $\sin x$ and 2 nd bracket is the expansion of $\cos x$, so we have $e^{ix} = \sin x + i \cos x$

So, $e^{i\theta} = (cos\theta + isin\theta)$ and

$e^{-i\theta} = (cos\theta - isin\theta)$

Euler forms make algebra very simple for complex numbers in cases where multiplication, division or powers of complex numbers are involved. Any complex number can be expressed as

$

\begin{aligned}

& \mathrm{z}=\mathrm{x}+\mathrm{iy} \

& \mathrm{z}=\mid \mathrm{z}(\cos \theta+\mathrm{i} \sin \theta) \

& \mathrm{z}=|\mathrm{z}| \mathrm{e}^{\mathrm{i} \theta}

\end{aligned}

$

(Cartesian form)

(Polar form)

(Eular's form)

Application of Euler form:

1. Multiplication of two complex numbers:

Let $\mathrm{z} = |\mathrm{z}|\mathrm{e}^{i\theta_1}$

And $w = |w|e^{i\theta_2}$

Multiplying these two number

$

\begin{aligned}

& \mathrm{z} \cdot \mathrm{w}=|\mathrm{z}| \mathrm{e}^{\mathrm{i} \theta_1} \cdot|\mathrm{w}| \mathrm{e}^{\mathrm{i} \theta_2} \

& =|\mathrm{z}| \cdot|\mathrm{w}| \mathrm{e}^{\mathrm{i}\left(\theta_1+\theta_2\right)}

\end{aligned}

$

2. Division also can be done in the same way,

$\mathrm{z}=|\mathrm{z}|\mathrm{e}^{\mathrm{i}\theta_1}$ and $\mathrm{w}=|\mathrm{w}|\mathrm{e}^{\mathrm{i}\theta_2}$ be two complex number

$

\therefore \frac{\mathrm{z}}{\mathrm{w}}=\frac{|\mathrm{z}|}{|\mathrm{w}|} \mathrm{e}^{\mathrm{i}\left(\theta_1-\theta_2\right)}

$

3. The logarithm of Complex Number

$$\mathrm{z}=|\mathrm{z}|\mathrm{e}^{\mathrm{i}\theta}$$
$$\log_{\mathrm{e}}(\mathrm{z})=\log_{\mathrm{e}}\left(|\mathrm{z}|\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$\log_{\mathrm{e}}(\mathrm{z})=\log_{\mathrm{e}}(|\mathrm{z}|)+\log_{\mathrm{e}}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$\log_{\mathrm{e}}(\mathrm{z})=\log_{\mathrm{e}}(|\mathrm{z}|)+\mathrm{i}\arg(\mathrm{z})$$

De Moivre's Theorem: $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$
Used for computing powers and roots of complex numbers in polar form.

Cube Roots of Unity: The cube roots of unity are:
$1,\omega,\omega^2$ where $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
They satisfy:
$1+\omega+\omega^2=0$ and $\omega^3=1$

Let z be the cube root of unity (1)

So, z_3 = 1

⇒ z_3 - 1 = 0

⇒ (z - 1)(z_2 + z + 1) = 0

⇒ z - 1 = 0 or z_2 + z + 1 = 0

$$

\therefore z=1 \text { or } z=\frac{-1 \pm \sqrt{(1-4)}}{2}=\frac{-1 \pm i \sqrt{3}}{2}

$$

Therefore, $\mathrm{z}=1,\mathrm{z}=\frac{-1+\mathrm{i}\sqrt{3}}{2}$ and $\mathrm{z}=\frac{-1-\mathrm{i}\sqrt{3}}{2}$

If the second root is represented by $\boldsymbol{\omega}$, then the third root will be represented by $\boldsymbol{\omega}_2$ (we can check that by squaring the second root, we get the third root)

$$
\omega=\frac{-1+\mathrm{i} \sqrt{3}}{2}, \quad \omega^2=\frac{-1-\mathrm{i} \sqrt{3}}{2}
$$

So, 1, ω, ω^2 are cube roots of unity and ω, ω^2 are the non-real complex root of unity.

Properties of Cube roots of unity

i) $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$ (Using sum and produc of roots relations for the equation $z_3 - 1 = 0$)

ii) To find ω^n , first we write ω in multiple of 3 with remainder being 0 or 1 or 2.

Now $\omega^n = \omega^{3q + r} = (\omega^3)^q \cdot \omega^r = \omega^r$ (Where r is from 0,1,2)

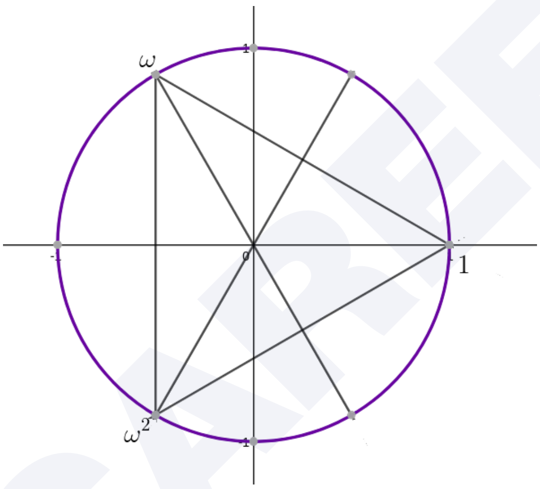
e.g., $\omega^{121} = \omega^{3.40 + 1} = (\omega^3)^{40} \cdot \omega^1 = \omega$

iii) $|\omega| = |\omega^2| = 1$, $\arg(\omega) = 2\pi/3$, $\arg(\omega^2) = 4\pi/3$ or $-2\pi/3$

iv) We can see that ω and ω^2 differ by the minus sign of imaginary part hence $\bar{\omega}=\omega^2$

v) Cube roots of -1 are -1, -ω, $-\omega^2$

vi) The cube roots of unity when represented on the complex plane has its point on vertices of triangle circumscribed by a unit circle whose one vertices lies on the +ve X-axis.



n-th Roots of Unity: Solutions of $z^n = 1$ are given by:
$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, 2, \ldots, n-1$

Distance Between Two Complex Numbers: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then
$|z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Equation of Perpendicular Bisector: The perpendicular bisector of the line segment joining z_1 and z_2 is:
$|z - z_1| = |z - z_2|$

Equation of Circle in Complex Plane: If center is z_0 and radius is r, the equation is:
$|z - z_0| = r$

Quadratic Equations in Complex Numbers:

For quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

the roots may be complex, given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relation Between Roots and Coefficients:

If roots are α and β, then:

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

Nature of Roots:

If $b^2 - 4ac > 0$, roots are real and distinct.

If $b^2 - 4ac = 0$, roots are real and equal.

If $b^2 - 4ac < 0$, roots are complex conjugates.

Formation of Quadratic Equation with Given Roots:

If roots are α and β, the quadratic equation is:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Chapter 5: Permutations and Combinations

Fundamental Principle of Counting: If an event can occur in m ways and another in n ways, then the total number of ways both can occur is:
$m \times n$

Permutation as an Arrangement: Number of ways to arrange r objects out of n:
$$P(n, r) = nPr = \frac{n!}{(n-r)!}$$

Permutation of Objects When Few are Identical: Number of permutations of n objects, where p, q, r, ... are identical:
$$\frac{n!}{p! \cdot q! \cdot r! \cdots}$$

Rank of a Word in Dictionary: Rank = (Number of words before it) + 1
Calculate by fixing letters from left to right and counting permutations of remaining letters.

Combination of Distinct Objects: Number of ways to choose r objects out of n (order does not matter):
$$C(n, r) = nCr = \frac{n!}{r!(n-r)!}$$

Selection of Identical Number of Objects: If repetition is allowed, the number of ways to select r items from n types:
$$C(n + r - 1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

Distribution of Distinct Objects into Distinct Places: If n distinct objects are to be distributed into r distinct boxes:
r^n ways

Distribution of Identical Objects into Distinct Places: Number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_r = n$:
$C(n+r-1, r-1)$

Derangement (No Object in Original Position): Number of derangements of n distinct items:
$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

Distribution of Distinct Objects into Identical Boxes (Unlabelled Boxes):

Number of ways to distribute n distinct items into r identical boxes depends on partitions — no direct formula, count partitions.

Distribution of Identical Objects into Identical Boxes:

Count the number of integer partitions of n into $\leq r$ parts.

Number of Ways to Arrange Objects in a Circle:

Without rotation considered: $(n-1)!$

If clockwise and counterclockwise are same: $\frac{(n-1)!}{2}$

Number of Permutations with Restrictions:

If certain items must be together: Treat them as one unit and multiply by their internal arrangements.

If certain items must not be together: Total permutations $-$ restricted permutations.

Number of Ways to Divide n Distinct Items into r Equal Groups:

If groups are distinguishable: $\frac{n!}{(k!)^r}$ (where each group has k elements, $n = kr$)

If groups are indistinguishable: Divide above by $r!$ → $\frac{n!}{r!(k!)^r}$

Number of Ways to Select Even/Odd Number of Objects from n:

Sum up $C(n, r)$ for even or odd values of r.

For even: $\sum_{r=0,2,4,\ldots}^{n} C(n,r) = 2^{n-1}$

For odd: $\sum_{r=1,3,5,\ldots}^{n} C(n,r) = 2^{n-1}$

Circular Permutations with Repetition Not Allowed:

If there are n people, and some positions are fixed, use:

$(n-1)!$ if no reference point

$(n-2)!$ if one person is fixed (e.g., host)

Number of Ways to Select At Least One Item: From n distinct items: $2^n - 1$

Chapter 6: Binomial Theorem

Factorials and nCr: $n! = n \times (n-1) \times (n-2) \cdots 1$
$nCr = \frac{n!}{r!(n-r)!}$

Binomial Theorem Expansion: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

General Term in Binomial Expansion: $T_{r+1} = \binom{n}{r} x^{n-r} y^r$

Middle Term(s): If n is even: $\text{Only one middle term} = T_{\frac{n}{2}+1}$
If n is odd: $\text{Two middle terms} = T_{\frac{n+1}{2}},\ T_{\frac{n+3}{2}}$

(p + 1)th Term from End: $(p+1)$th term from end = $(n-p+1)$th term from beginning

Greatest Term in $(x+y)^n$ (for $x = y = 1$): $r = \left\lfloor \frac{n+1}{2} \right\rfloor$
Greatest term = $T_r = \binom{n}{r-1}$

Standard Expansions: $(1+x)^n, \quad (1-x)^n, \quad (x-y)^n, \quad (x+a)^n$
use same binomial formula with proper signs

Differentiation Form of Binomial Coefficients: $\sum_{r=1}^{n} r \cdot \binom{n}{r} = n \cdot 2^{n-1}$
$\sum_{r=0}^{n} r^2 \cdot \binom{n}{r} = n(n+1) \cdot 2^{n-2}$

Product of Two Binomial Coefficients: $\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}$

Multinomial Theorem: $(x_1 + x_2 + \cdots + x_k)^n = \sum \frac{n!}{r_1! r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$
where $r_1 + r_2 + \cdots + r_k = n$

Chapter 7: Sequence & Series

Arithmetic Progression (A.P.): For a as first term and d as common difference,

nth term:
$a_n = a + (n-1)d$
Sum of first n terms:
$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$

Geometric Progression (G.P.): nth term:
$a_n = ar^{n-1}$
Sum of first n terms ($r \neq 1$):
$S_n = \frac{a(r^n - 1)}{r-1}$
Sum to infinity ($|r| < 1$):
$S_\infty = \frac{a}{1-r}$

Insertion of A.M.s and G.M.s Between Two Numbers:
To insert n A.M.s between a and b:
Common difference $d = \frac{b-a}{n+1}$
To insert n G.M.s between a and b:
Common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Relation Between A.M. and G.M.:
If a and b are two positive numbers:
$\text{A.M.} = \frac{a+b}{2}, \quad \text{G.M.} = \sqrt{ab}$
Then: $\text{A.M.} \geq \text{G.M.}$

Harmonic Progression (H.P.):
If $a_1, a_2, \ldots$ is an H.P., then $\frac{1}{a_1}, \frac{1}{a_2}, \ldots$ form an A.P.

Harmonic Mean (H.M.):
Between a and b:
$\text{H.M.} = \frac{2ab}{a+b}$

Sum of Special Series:

Sum of First n Natural Numbers (Method of VN):
$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Sum of First n Squares:
$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of First n Cubes:
$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

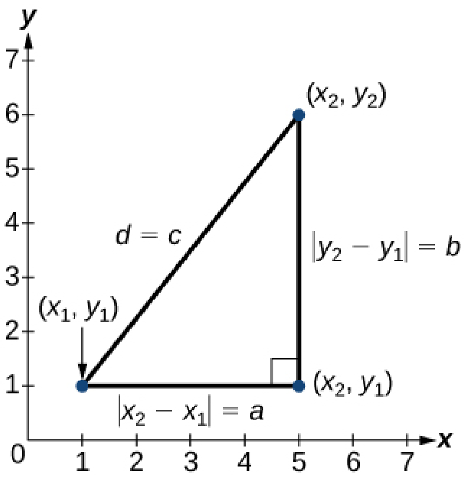
Method of Differences:
If $S = \sum_{k=1}^{n}(a_k - a_{k+1})$, then
$S = a_1 - a_{n+1}$

Sum of Infinite G.P. (where $|r| < 1$):
$S_\infty = \frac{a}{1-r}$

Arithmetic-Geometric Progression (A.G.P.):
For series $S = a + (a+d)r + (a+2d)r^2 + \cdots + (a+(n-1)d)r^{n-1}$,
Sum:
$S_n = \frac{a(1-r^n)}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}$

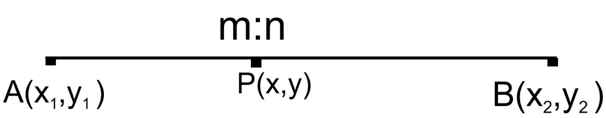
Chapter 8: Straight Lines

Distance Between Two Points:



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Section Formula (Internal division):



$$\mathbf{x} = \frac{\mathbf{mx_2 + nx_1}}{\mathbf{m+n}}, \quad \mathbf{y} = \frac{\mathbf{my_2 + ny_1}}{\mathbf{m+n}}$$

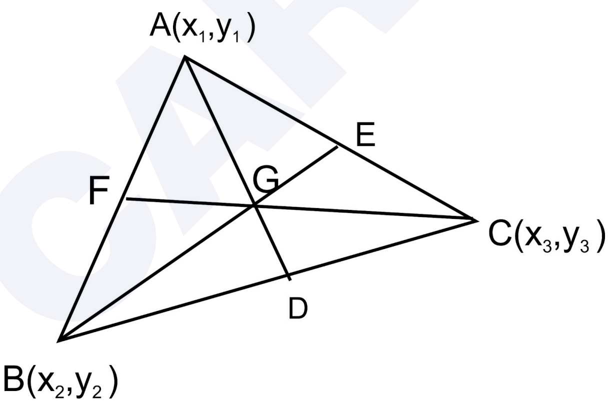
$$P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

Section Formula (External division):

$$\mathbf{x} = \frac{\mathbf{mx_2 - nx_1}}{\mathbf{m-n}}, \quad \mathbf{y} = \frac{\mathbf{my_2 - ny_1}}{\mathbf{m-n}}$$

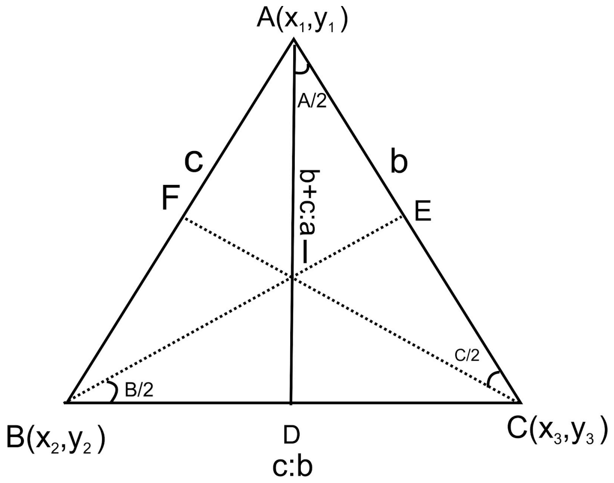
$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

Centroid of Triangle:



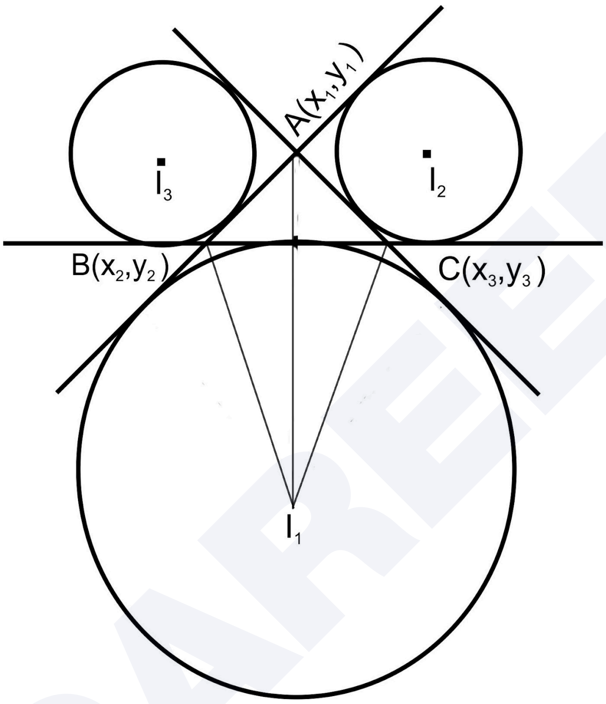
$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Incentre of Triangle:



$$I=\left(\frac{a x_1+b x_2+c x_3}{a+b+c}, \frac{a y_1+b y_2+c y_3}{a+b+c}\right)$$

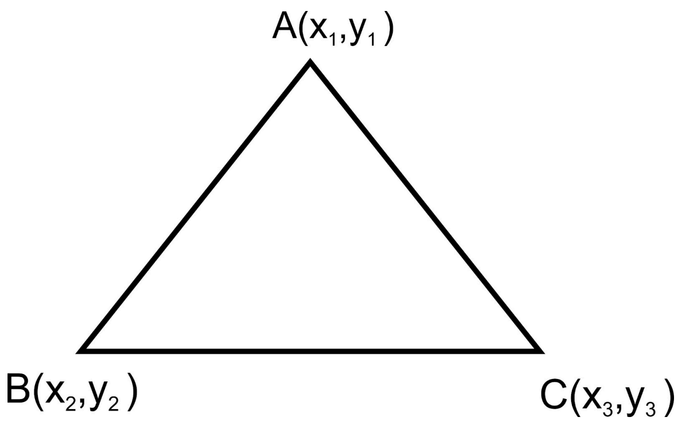
Excentre of Triangle (Example for E_a):



$$E_a=\left(\frac{-a x_1+b x_2+c x_3}{-a+b+c}, \frac{-a y_1+b y_2+c y_3}{-a+b+c}\right)$$

Area of Triangle:

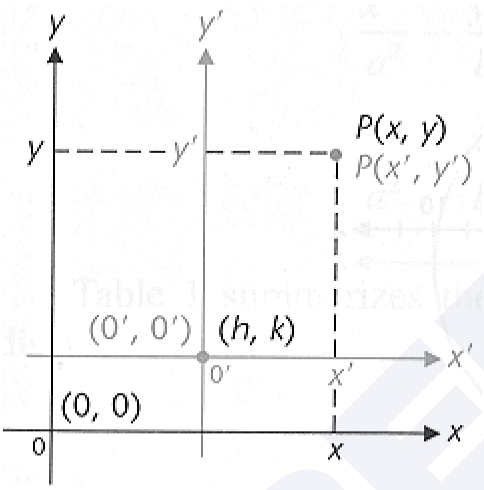
If vertices of a triangle ABC given as A (x_1, y_1), B (x_2, y_2) and C(x_3, y_3), then area of ΔABC is



$$\text{Area} = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Locus and Its Equation: $f(x, y) = 0$

Transformation of Axes (Translation):

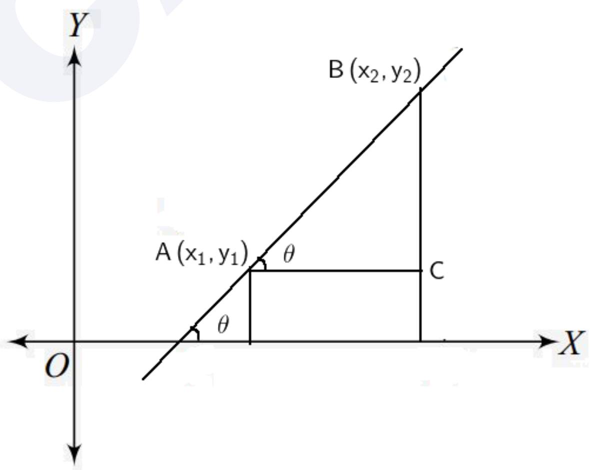


$$X = x - h, \quad Y = y - k$$

Transformation of Axes (Rotation):

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta$$

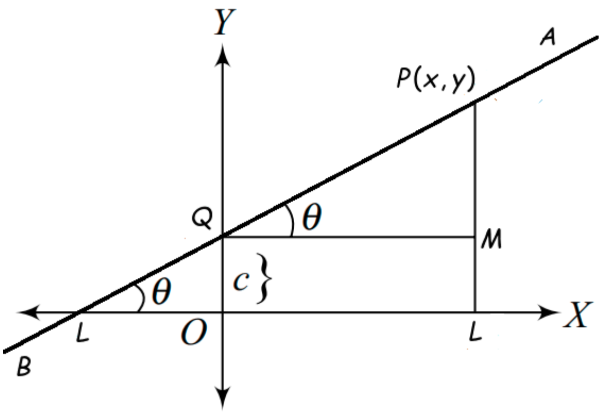
Slope of a Line:



$m=\frac{y_2-y_1}{x_2-x_1}$

Point-Slope Form: $y-y_1=m(x-x_1)$

Slope-Intercept Form:



$y=mx+c$

General Form of Line: $Ax+By+C=0$

Normal form of line

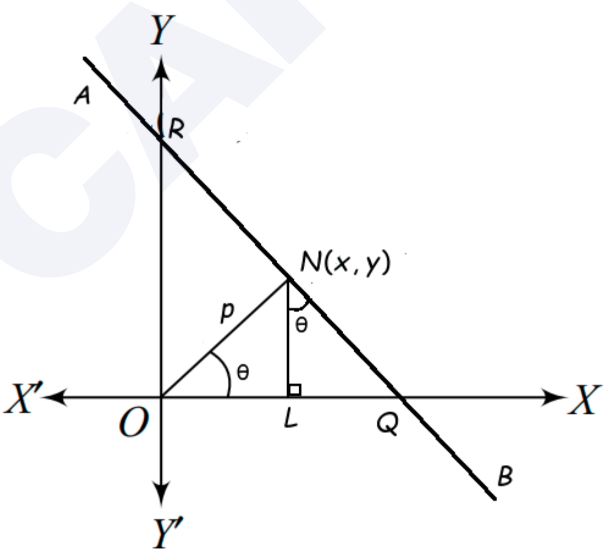
Equation of straight line on which the length of the perpendicular from the origin is p and this normal makes an angle θ with the positive direction of X-axis is given by

$\mathbf{x}\cos\theta+\mathbf{y}\sin\theta=\mathbf{p}$

Proof:

AB is the straight line and length of perpendicular from origin to the line is p (i.e. ON = p).

Line AB cuts X-axis and Y-axis at point Q and R respectively



$$

\begin{aligned}

& \quad \angle \mathrm{NOX}=\theta \

& \angle \mathrm{NQO}=90^{\circ}-\theta \

& \therefore \angle \mathrm{NQX}=180^{\circ}-\left(90^{\circ}-\theta\right)=90^{\circ}+\theta \

& \text { Slope } m=\tan \left(90^{\circ}+\theta\right)=-\cot (\theta)

\end{aligned}

$$

In triangle NOL

$\begin{aligned}

& O L=x=p \cdot \cos \theta, N L=y=p \cdot \sin \theta \

& \text { Point } N(p \cdot \cos \theta, p \cdot \sin \theta)

\end{aligned}$

Using Slope-point form, equation of line AB is

$\begin{aligned}

& y-p \cdot \sin \theta=-\frac{\cos \theta}{\sin \theta}(x-p \cdot \cos \theta) \

& x \cdot \cos \theta+y \cdot \sin \theta=p

\end{aligned}$

Parametric form of a line

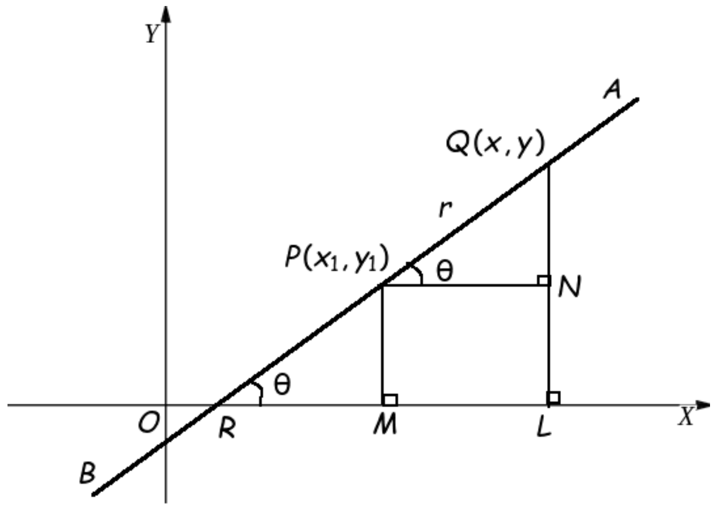
The equation of a straight line passing through the point (x_1,y_1) and making an angle θ with the positive direction of X-axis is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

Where r is the directed distance between the points (x, y) and (x_1,y_1).

Proof:

AB is a straight line passing through the point P(x_1,y_1) and meets X-axis at R and makes an angle θ with the positive direction of X-axis.



Let Q(x, y) be any point on the line AB at a distance 'r' from P

As from the figure

$\begin{aligned}

& \mathrm{PN}=\mathrm{ML}=\mathrm{OL}-\mathrm{OM}=\mathrm{x}-\mathrm{x}_1 \

& \mathrm{QN}=\mathrm{QL}-\mathrm{NL}=\mathrm{QL}-\mathrm{PM}=\mathrm{y}-\mathrm{y}_1 \

& \mathrm{In} \Delta \mathrm{NPQ} \

& \cos \theta=\frac{\mathrm{PN}}{\mathrm{PQ}}=\frac{\mathrm{x}-\mathrm{x}_1}{\mathrm{r}} \

& \sin \theta=\frac{\mathrm{QN}}{\mathrm{PQ}}=\frac{\mathrm{y}-\mathrm{y}_1}{\mathrm{r}}

\end{aligned}$

From the above two equation

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$$

Also,

$\begin{aligned}

& \mathrm{x}=\mathrm{x}_1+\mathrm{r} \cos \theta \

& \mathrm{y}=\mathrm{y}_1+\mathrm{r} \sin \theta

\end{aligned}$

Parametric equations of straight line AB

Angle Between Two Lines: $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

Distance of a Point from a Line: $d = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$

Point of Intersection of Two Lines:

$$A_1x + B_1y + C_1 = 0$$

$A_2x + B_2y + C_2 = 0$

Family of Lines: $L_1 + \lambda L_2 = 0$

Equation of Bisectors of Angles Between Two Lines:

$$\frac{L_1}{\sqrt{A_1^2+B_1^2}} = \pm\frac{L_2}{\sqrt{A_2^2+B_2^2}}$$

Foot of Perpendicular from (x_0, y_0) to Line $Ax + By + C = 0$:

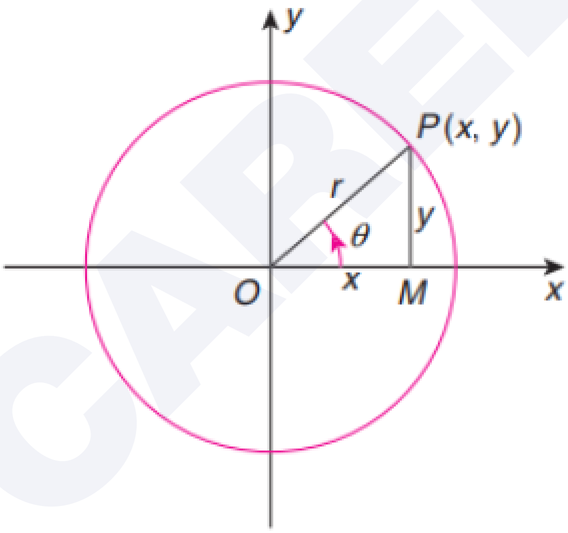
$$\left(\frac{B^2x_0-ABy_0-AC}{A^2+B^2}, \frac{A^2y_0-ABx_0-BC}{A^2+B^2}\right)$$

Image of a Point About a Line: $(2x_f - x_0, 2y_f - y_0)$

Chapter 9: Conic Sections

Diameter Form of a Circle: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Parametric Form of a Circle (center (h, k) and radius r):



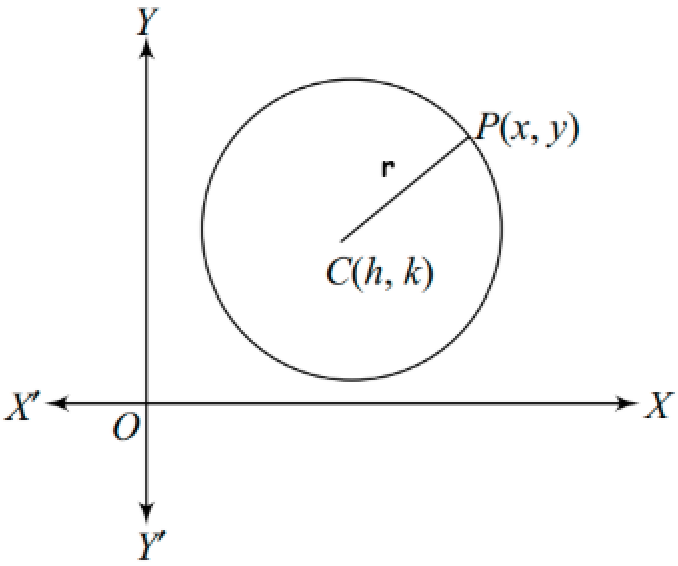
$x = h + r\cos\theta, \quad y = k + r\sin\theta$

Intercepts Made by Circle on Axes:

x-intercepts: Set $y = 0$ in circle equation

y-intercepts: Set $x = 0$ in circle equation

Position of a Point (x_0, y_0) w.r.t Circle $(x - h)^2 + (y - k)^2 = r^2$:



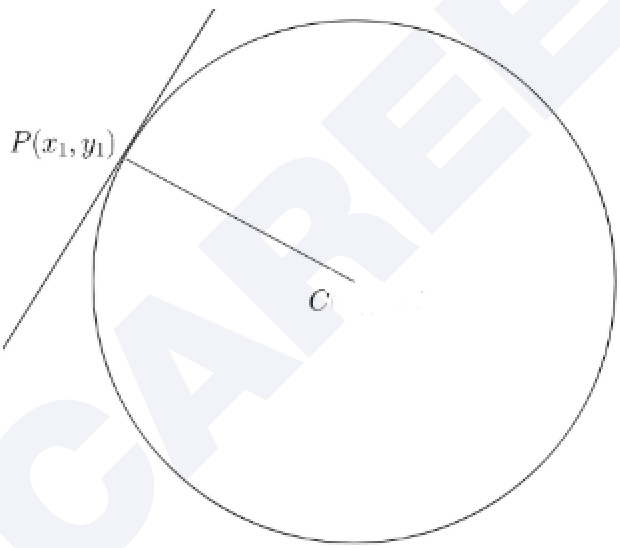
Calculate $d = (x_0 - h)^2 + (y_0 - k)^2$

If $d < r^2$, point lies inside

If $d = r^2$, point lies on the circle

If $d > r^2$, point lies outside

Equation of Tangent to Circle at Point (x_1, y_1) on Circle:



$(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2$

Equation of Tangent in Parametric Form (θ is parameter):

$x \cos\theta + y \sin\theta = r$

Equation of Tangent with Slope m:

$y = mx \pm r\sqrt{1 + m^2}$

Chord of Contact of Tangents from Point (x_1, y_1) to Circle:

$T = S_1$ where

$S = (x-h)^2 + (y-k)^2 - r^2$,

$T = (x-h)(x_1-h) + (y-k)(y_1-k) - r^2$,

$S_1 = (x_1-h)^2 + (y_1-k)^2 - r^2$

Equation of Normal to Circle at (x_1, y_1):

Line perpendicular to tangent, slope of normal is negative reciprocal of tangent's slope.

Director Circle of Circle $(x-h)^2 + (y-k)^2 = r^2$:

$(x-h)^2 + (y-k)^2 = 2r^2$

Angle of Intersection of Two Circles:

If radii are r_1, r_2 and distance between centers is d, then

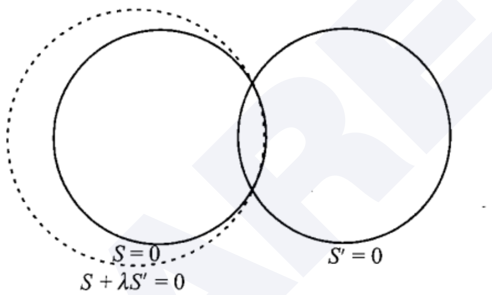
$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

Family of Circles Passing Through Intersection of Two Circles $S_1 = 0$ and $S_2 = 0$:

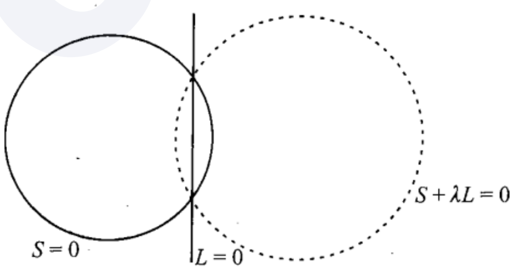
$S_1 + \lambda S_2 = 0$

Family of Circles

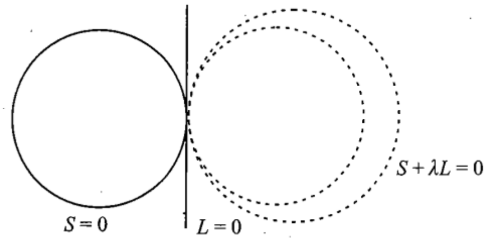
1. Equation of the family of circles passing through the point of intersection of two given circles S = 0 and S' = 0 is S + λS' = 0 where, λ is the parameter



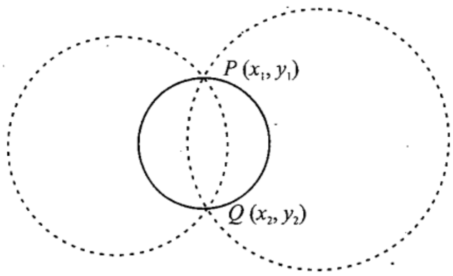
2. Equation of the family of circles passing through the point of intersection of a given circle S = 0 and a line L = 0is S + λL = 0 where, λ is the parameter.



3. Equation of the family of circles touching the given circle S = 0 and the line L = 0 is S + λL=0



4. Equation of the family of circles passing through the two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is



$\left(\mathrm{x}-\mathrm{x}_1\right)\left(\mathrm{x}-\mathrm{x}_2\right)+\left(\mathrm{y}-\mathrm{y}_1\right)\left(\mathrm{y}-\mathrm{y}_2\right)+\lambda\left|\begin{array}{lll}

x & y & 1 \

x_1 & y_1 & 1 \

x_2 & y_2 & 1

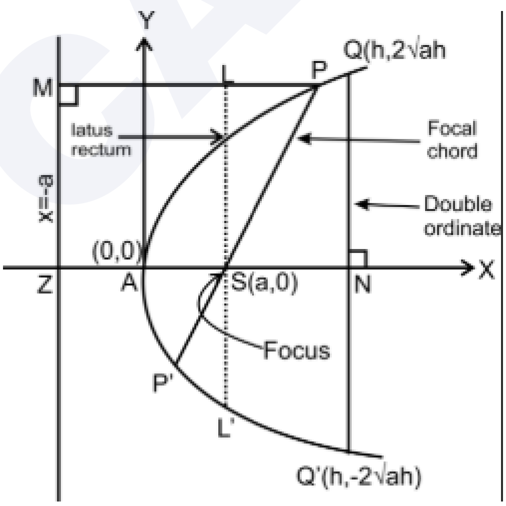
\end{array}\right|=0$

5. Equation of the family of circles which touch $\mathrm{y}-\mathrm{y}_1=\mathrm{m}\left(\mathrm{x}-\mathrm{x}_1\right)$ at $\left(\mathrm{x}_1, \mathrm{y}_1\right)$ for any finite m is $\left(x-x_1\right)^2+\left(y-y_1\right)^2+\lambda\left\{\left(y-y_1\right)-m\left(x-x_1\right)\right\}=0$

And if m is infinite then the family of circles is $\left(x-x_1\right)^2+\left(y-y_1\right)^2+\lambda\left(x-x_1\right)=0$

Radical Axis of Two Circles $S_1=0$ and $S_2=0$: $S_1-S_2=0$

Parabola (Standard Form - Vertex at Origin):



$y^2=4ax$: Focus: $(a, 0)$, Directrix: $x=-a$, Axis: x-axis

$x^2 = 4ay$: Focus: $(0, a)$, Directrix: $y = -a$, Axis: y-axis

Length of latus rectum = $4a$

Parabola (General Form - Vertex at (h, k)):

$(y-k)^2 = 4a(x-h)$ or $(x-h)^2 = 4a(y-k)$

Focus: $(h+a, k)$ or $(h, k+a)$

Directrix: $x = h - a$ or $y = k - a$

Length of latus rectum = $4a$

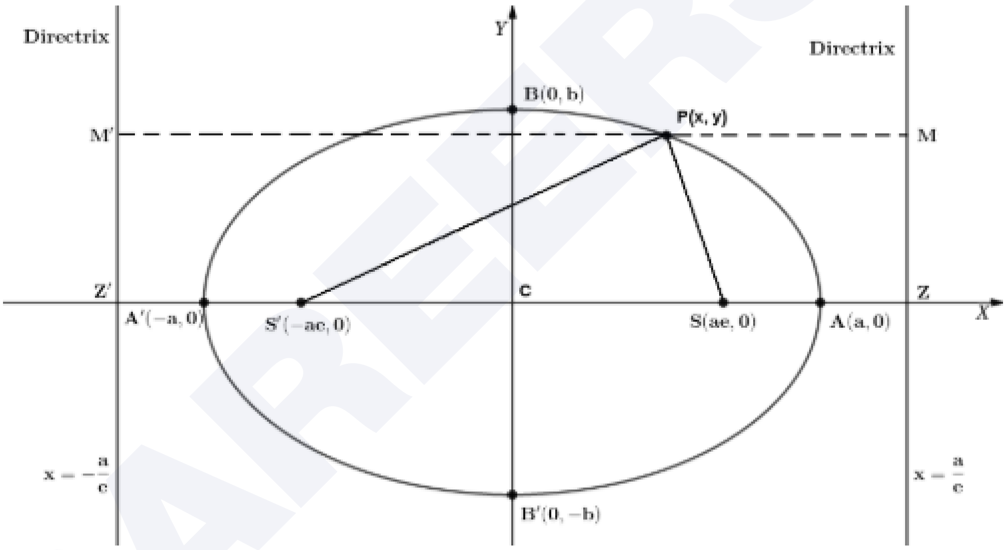
Parametric Coordinates on Parabola:

On $y^2 = 4ax$: $(at^2, 2at)$

Tangent: $yy_1 = 2a(x + x_1)$

Normal: $y = -tx + 2at + at^3$

Ellipse (Standard Form - Center at Origin):



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$

Foci: $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$

Directrices: $x = \pm \frac{a^2}{c}$

Eccentricity: $e = \frac{c}{a}$

Length of major axis: $2a$, Length of minor axis: $2b$

Latus rectum = $\frac{2b^2}{a}$

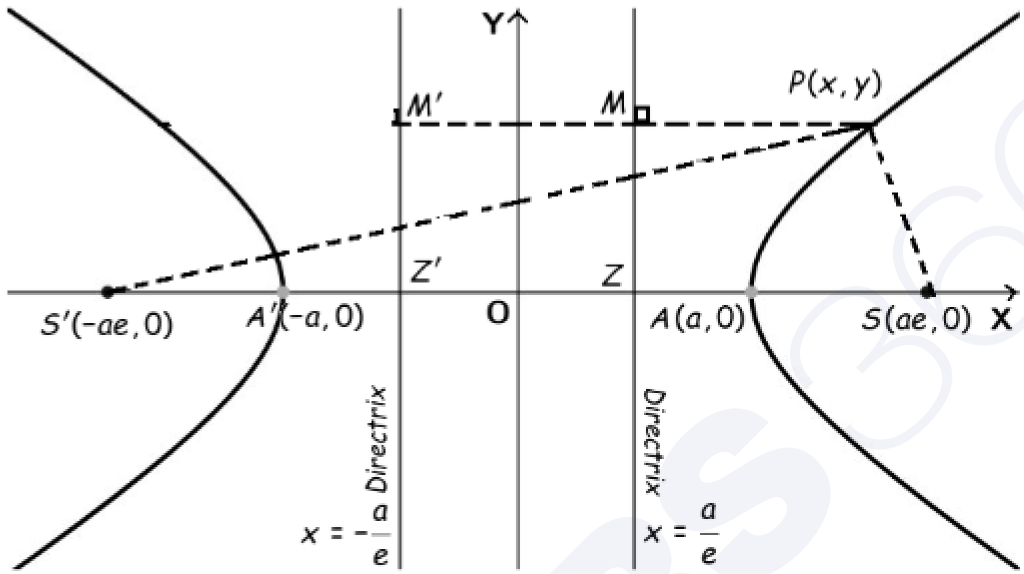
Ellipse (Parametric Form):

$x = a\cos\theta, y = b\sin\theta$

Ellipse (General Form - Center at (h, k)):

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Hyperbola (Standard Form - Center at Origin):



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci: $(\pm c, 0)$, where $c = \sqrt{a^2 + b^2}$

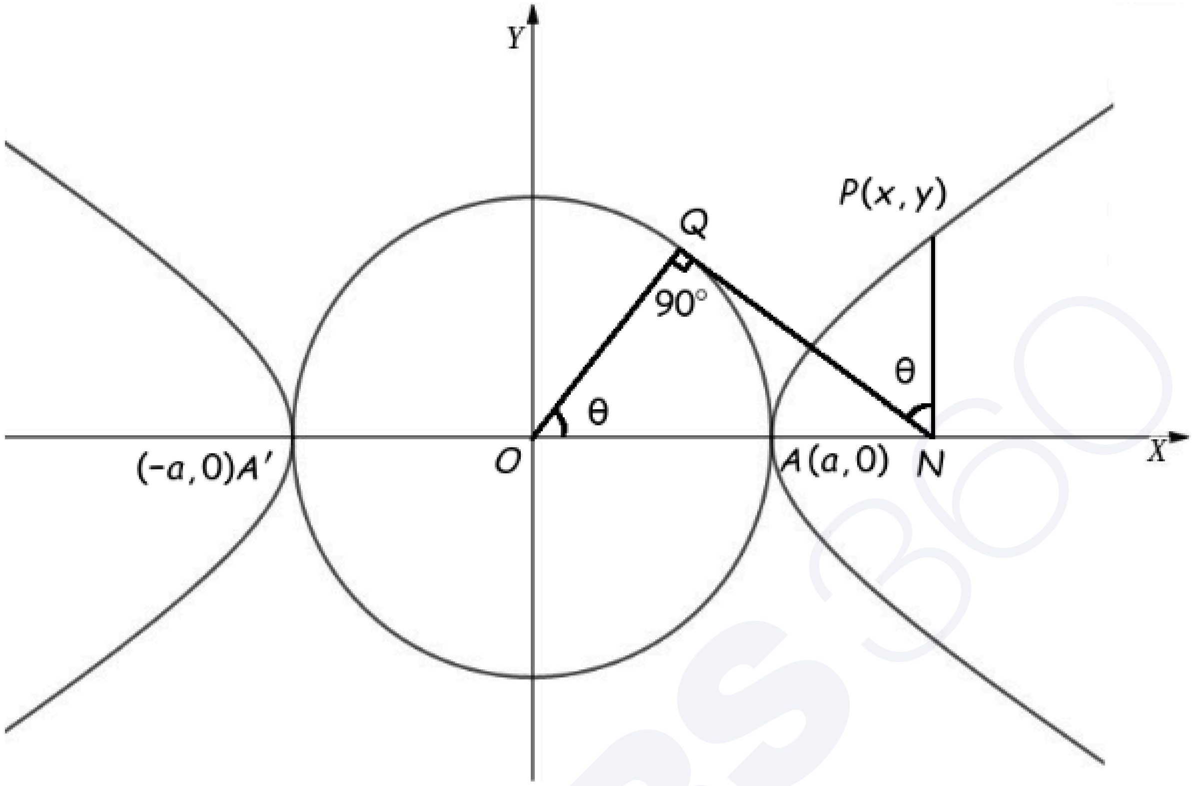
Directrices: $x = \pm \frac{a^2}{c}$

Eccentricity: $e = \frac{c}{a}$

Latus rectum = $\frac{2b^2}{a}$

Asymptotes: $y = \pm \frac{b}{a} x$

Hyperbola (Parametric Form):



$x = a\sec\theta, y = b\tan\theta$

Hyperbola (Second Type):

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

General Equation of a Conic:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Use discriminant $\Delta = B^2 - 4AC$:

If $\Delta = 0$: Parabola

If $\Delta < 0$: Ellipse

If $\Delta > 0$: Hyperbola

Chapter 10: 3D Geometry

Section Formula (3D): The point P divides the line into m:n

Internal Division: $P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

External Division: $P = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$

Equation of a Line (Vector Form): $\vec{r}=\vec{a}+\lambda\vec{b}$

where $\vec{a}$ is a point on the line and $\vec{b}$ is direction vector.

Equation of a Line (Cartesian Form):

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

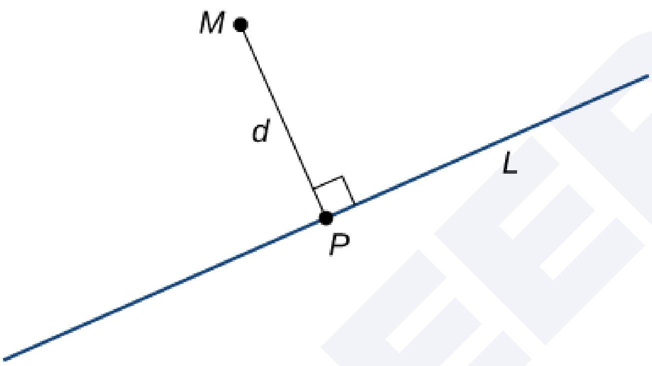
where (l, m, n) are direction ratios.

Angle Between Two Lines:

$\cos\theta=\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

Perpendicular Distance from Point to Line:

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of the line segment $\overrightarrow{MP}$, where P is a point on L such that $\overrightarrow{MP}$ is perpendicular to L.



Cartesian Form

The equation of line L is $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$

Let P be the foot of perpendicular drawn from the point $M(\alpha,\beta,\gamma)$ on the line L.

Let the coordinates of P be $(x_0+a\lambda, y_0+b\lambda, z_0+c\lambda)$

Then the direction ratio of MP are $(x_0+a\lambda-\alpha, y_0+b\lambda-\beta, z_0+c\lambda-\gamma)$.

Direction ratio of line L are (a, b, c)

Since, MP is perpendicular to line L,

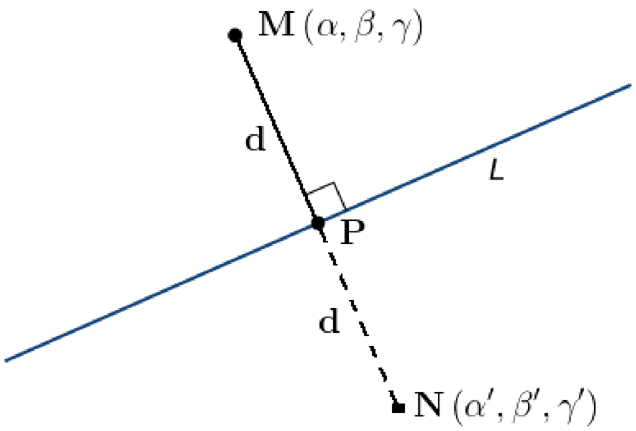
$a\left(x_0+a\lambda-\alpha\right)+b\left(y_0+b\lambda-\beta\right)+c\left(z_0+c\lambda-\gamma\right)=0$

$\lambda=\frac{a(\alpha-x_0)+b(\beta-y_0)+c(\gamma-z_0)}{a^2+b^2+c^2}$

Put the value of λ in $(x_0+a\lambda, y_0+b\lambda, z_0+c\lambda)$.

we get the foot of the perpendicular. Now, we can get distance MP using distance formula.

Image of a point



Since P (foot of perpendicular) is the midpoint of M and N (image of a point M in the line), we can get N if P is found out.

Cartesian Form

Let M(α, β, γ) be the point and $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ be the equation of line L.

Let P be the foot of perpendicular from M to line L and let N be the image of the point in the given line, where MP = PN

Let the coordinates of P be

$$\left(x_0+a\lambda, y_0+b\lambda, z_0+c\lambda\right)$$

Then, direction ratios of PL are

$$\left(x_0+a\lambda-\alpha, y_0+b\lambda-\beta, z_0+c\lambda-\gamma\right)$$

since, MP is perpendicular to the given line, whose direction ratios are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

$$\therefore a\cdot\left(x_0+a\lambda-\alpha\right)+b\cdot\left(y_0+b\lambda-\beta\right)+c\cdot\left(z_0+c\lambda-\gamma\right)=0$$

$$\Rightarrow \lambda=\frac{a\left(\alpha-x_0\right)+b\left(\beta-y_0\right)+c\left(\gamma-z_0\right)}{a^2+b^2+c^2}$$

Substituting the value of λ we get coordinates of point **P** (foot of perpendicular)

As **N (α', β', γ')** is image of point **M(α, β, γ)**

∴ mid-point of **MN** is point **P**

$$\begin{aligned} &\therefore \quad \frac{\alpha+\alpha'}{2}=x_0+a\lambda, \quad \frac{\beta+\beta'}{2}=y_0+b\lambda, \quad \frac{\gamma+\gamma'}{2}=z_0+\alpha\lambda \\ &\therefore \quad \alpha'=2\left(x_0+a\lambda\right)-\alpha, \beta'=2\left(y_0+b\lambda\right)-\beta, \gamma'=2\left(z_0+c\lambda\right)-\gamma \end{aligned}$$

Shortest Distance Between Two Skew Lines:

$$D=\frac{\left|\left(\vec{a_2}-\vec{a_1}\right)\cdot\left(\vec{b_1}\times\vec{b_2}\right)\right|}{\left|\vec{b_1}\times\vec{b_2}\right|}$$

Equation of a Plane in Normal Form:

$$\vec{r}\cdot\hat{n}=p$$

where $\hat{n}$ is a unit normal vector and p is perpendicular from origin.

Plane Perpendicular to Vector $\vec{n}$ Through Point $\vec{a}$:

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

Plane Through Three Non-Collinear Points:

Use determinant:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

Plane Through Point and Parallel to Vectors $\vec{b_1},\vec{b_2}$:

$$(\vec{r}-\vec{a})\cdot(\vec{b_1}\times\vec{b_2})=0$$

Distance of a Point (x_1,y_1,z_1) from Plane $ax+by+cz+d=0$:

$$D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Coplanarity of Two Lines: Lines are coplanar if

$$[(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})]=0$$

Chapter 11: Limits and Derivatives

Limits of a Function: $\lim_{x\to a} f(x)=L$

Left Hand Limit (LHL): $\lim_{x\to a^-} f(x)$

Right Hand Limit (RHL): $\lim_{x\to a^+} f(x)$

Algebra of Limits:

$$\lim_{x\to a}[f(x)\pm g(x)]=\lim f(x)\pm\lim g(x)$$

$$\lim_{x\to a}[f(x)\cdot g(x)]=\lim f(x)\cdot\lim g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim f(x)}{\lim g(x)}$$ (if $\lim g(x)\neq 0$)

Indeterminate Forms:

$$\frac{0}{0},\frac{\infty}{\infty},0\cdot\infty,\infty-\infty,0^0,\infty^0,1^\infty$$

Limits at Infinity:

$$\lim_{x\to\infty}\frac{1}{x}=0,\quad \lim_{x\to\infty}x^n=\infty$$

Limit Using Expansion:

$\lim_{x\to 0} \frac{e^x-1}{x} = 1, \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

Trigonometric Limits:

$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{\tan x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

Exponential Limits: $\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$

Logarithmic Limits: $\lim_{x\to 1} \frac{\ln x}{x-1} = 1$

L'Hôpital's Rule: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Limit Based on 1^∞: $\lim_{x\to 0}(1+x)^{1/x} = e$

Sandwich Theorem: If $g(x) \le f(x) \le h(x)$ and $\lim g(x) = \lim h(x) = L$, then

$\lim f(x) = L$

Continuity at $x = a$:

$f(x)$ is continuous at $x = a$ if

$\lim_{x\to a} f(x) = f(a)$

Discontinuity:

If any of LHL, RHL, or $f(a)$ is not defined or unequal, $f(x)$ is discontinuous at $x = a$

Intermediate Value Theorem:

If $f(x)$ is continuous on $[a, b]$, and $f(a) < k < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$

Derivatives of Basic Functions:

$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \sec^2 x$

Sum/Difference Rule: $(f \pm g)' = f' \pm g'$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain Rule:

If $y = f(u), u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Derivatives of Inverse Trigonometric Functions:

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Differentiation in Parametric Form:

If $x = f(t), y = g(t)$ then

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Derivative of Inverse Function:

If $y = f^{-1}(x)$, then

$\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$

Chapter 12: Statistics

-

Central Values:

Mean (Ungrouped Data): $\bar{x} = \frac{\sum x_i}{n}$

Mean (Grouped Data): $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

Median (Ungrouped Data):

Arrange data in ascending order. Median is middle value.

If even number of values, take average of two middle values.

Median (Grouped Data):

$$M = l + \left(\frac{\frac{N}{2} - F}{f}\right) \cdot h$$

Where:

l = lower boundary of median class

N = total frequency

F = cumulative frequency before median class

f = frequency of median class

h = class width

Mode (Grouped Data):

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \cdot h$$

Where:

l = lower boundary of modal class

f_1 = frequency of modal class

f_0 = frequency of class before modal class

f_2 = frequency of class after modal class

h = class width

Range:

$$\text{Range} = \text{Maximum value} - \text{Minimum value}$$

Mean Deviation (Ungrouped Data): $\text{M.D. about mean} = \frac{\sum |x_i - \bar{x}|}{n}$

Mean Deviation (Grouped Data): $\text{M.D. about mean} = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$

Variance (Ungrouped Data): $\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$

Standard Deviation (Ungrouped Data): $\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$

Variance (Grouped Data): $\sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}$

Standard Deviation (Grouped Data): $\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$

Chapter 13: Probability

Compound Event

If an event has more than one sample point, it is called a Compound event.

Mutually Exclusive Events
Two or more than two events are said to be mutually exclusive if the occurrence of one of the events excludes the occurrence of the other

Independent Events
Events can be said to be independent if the occurrence or non-occurrence of one event does not influence the occurrence or non-occurrence of the other.

Simple Event

If an event has only one sample point of a sample space, it is called a simple (or elementary) event.

Algebra of Events:

$P(S) = 1$, where S is the sample space

$0 \leq P(E) \leq 1$ for any event E

$P(\phi) = 0$ (Probability of impossible event)

$P(A') = 1 - P(A)$ (Complementary event)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Set Theory Notations in Probability:

$A \cup B$: Event A or B occurs

$A \cap B$: Both A and B occur

A': Event A does not occur

$A \subset B$: If A occurs, B must occur

Mutually exclusive: $A \cap B = \phi$

Exhaustive: $A \cup B \cup C \cup \cdots = S$

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Class 12

Chapter 1: Relations & Functions

Number of elements in A x B

If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$,

i.e., if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$

Number of Relations from A to B

If A have m elements and B have n elements, then $A \times B$ has $m \times n$ element.

As the number of subsets of $A \times B$ is 2^{mn}, and a relation is a subset of $A \times B$, so the total number of relations from A to B will be 2^{mn}.

Domain

The domain of a relation R is the set of all first elements of the ordered pairs in a relation R.

eg. R={(a,b),(c,d)}, then the domain is {a,c}

Range

The range of a relation R is the set of all second elements of the ordered pairs in a relation R.

eg. R={(a,b),(c,d)}. Then Range is {b,d}

Co-domain

Co-domain in a relation from A to B is the set B itself.

Reflexive Relation

- A relation R on a set a is **reflexive** if every element is related to itself.
- For all $a \in A$,
  $(a, a) \in R$

Symmetric Relation

- A relation R on a set a is **symmetric** if whenever an element a is related to b, then b is related to a.
- For all $a, b \in A$,
  $(a, b) \in R \implies (b, a) \in R$

Transitive Relation

- A relation R on a set a is **transitive** if whenever an element a is related to b, and b is related to c, then a is related to c.

- For all $a, b, c \in A$,
  $(a, b) \in R \text{ and } (b, c) \in R \implies (a, c) \in R$

Equivalence Relation

For all $a, b, c \in A$,

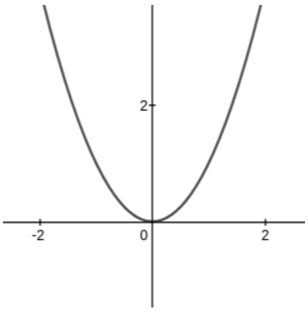
$$\begin{cases} (a, a) \in R & \text{(Reflexive)} \\ (a, b) \in R \implies (b, a) \in R & \text{(Symmetric)} \\ (a, b) \in R \text{ and } (b, c) \in R \implies (a, c) \in R & \text{(Transitive)} \end{cases}$$

Function: A function f from set A to set B assigns each element in A exactly one element in B.

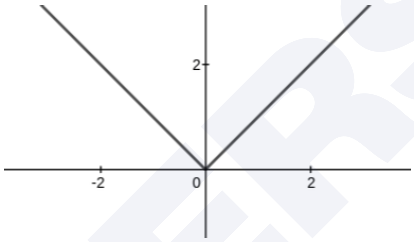
$f : A \to B$

Even function:

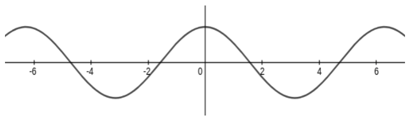
If for a function f(x), f(-x) = f(x) then the function is known as even function. Even functions are symmetric about the y axis.



$y = x^2$



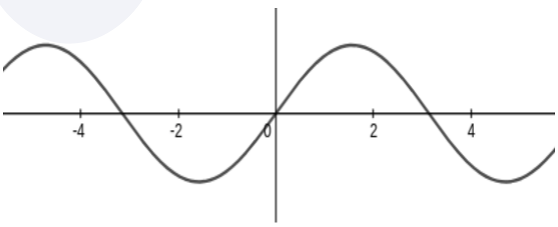
y = |x|



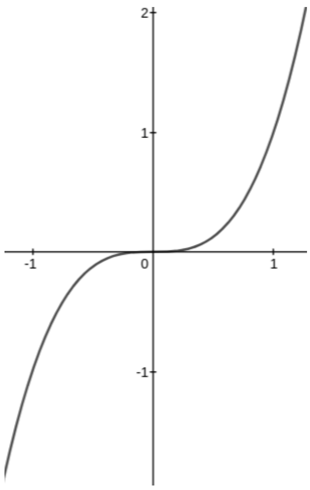
y = cos(x)

Odd function:

If for a function f(x), f(-x) = - f(x) then the function is known as odd function. Odd functions are symmetric about the origin.



y = sin(x)



$y = x^3$

NOTE: We can have functions that are neither even nor odd. e.g., y = x+1

Periodic Function

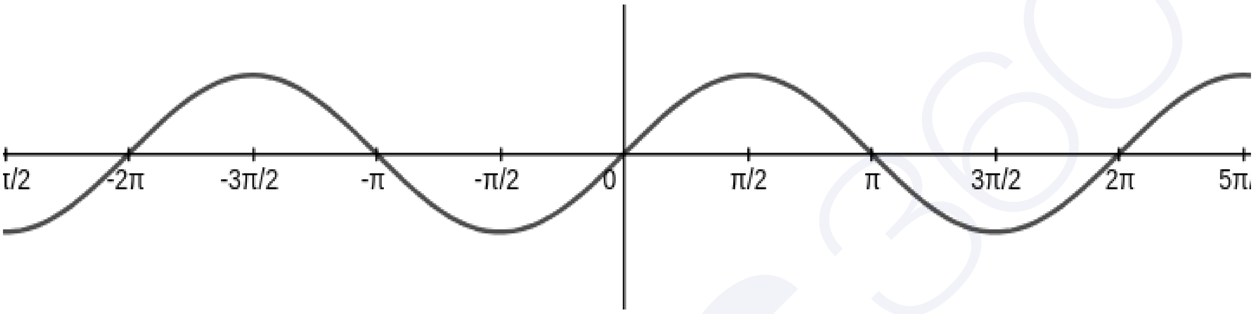
A function f(x) is called a periodic function, if there exists a +ve real number T such that f(x+T) = f(x) ∀ x ? Domain of f(x).

Here, T is called the period of f(x), where T is least +ve value.

Graphically: if the graph repeats at a fixed interval, the function is said to be periodic and its period is the width of that interval.

Eg

Graph of sin(x) is repeated at an interval of 2π



Some standard results

Functions	Period
sin(x), cos(x), sec(x), cosec(x)	2π
$\sin^2(x)$, $\cos^2(x)$	π
tan(x), cot(x)	π
|sin x|, |cos x|, |tan x|, |cot x|, |cosec x|, |sec x|	π
{x}	1
Algebric function, eg. x^2, $x^3 + 6$	Not Periodic

Properties of the periodic function

i) if f(x) is periodic with period T, then

1. cf(x) is periodic with period T
2. f(x+c) is periodic with period T
3. f(x) ± c is periodic with period T, where c is any constant.

ii) if f(x) is periodic with period T, then kf(cx+d) has period $\frac{T}{|c|}$ i.e. period is only affected by the coefficient of x,

iii) if $f_1(x), f_2(x)$ are periodic functions with periods T_1, T_2 respectively, then $h(x) = f_1(x) + f_2(x)$ has period

a). LCM of { T_1, T_2 }, if h(x) is not an even function.

or

b) 0.5 LCM of { T_1, T_2}, if $f_1(x)$ and $f_2(x)$ are complementary pairwise comparable functions.

Domain

All possible values of x for f(x) is defined (f(x) is a real number) is known as domain.

If a function is defined from A to B i.e. f: A→B, then all the elements of set A is called Domain of the function.

Co-domain

If a function is defined from A to B i.e. f: A→B, then set B is called Co-domain of the function.

Range

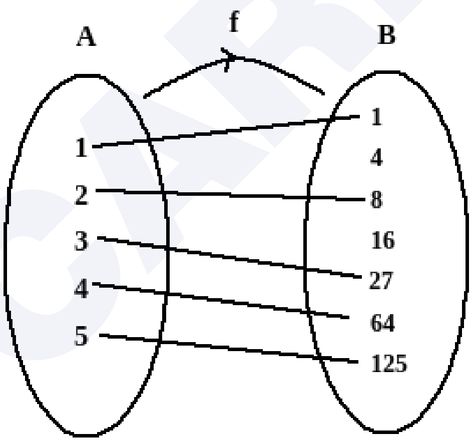
The set of all possible values of f(x) for every x belonging to the domain is known as Range of this function.

For example, let A = {1, 2, 3, 4, 5} and B = {1, 4, 8, 16, 27, 64, 125}. The function f : A -> B is defined by f(x) = x^3. So here,

Domain : Set A

Co-Domain : Set B

Range : {1, 8, 27, 64, 125}



The range is always a subset of co-domain and Range can be equal to co-domain in some cases.

Note: If only the formula is given, then co-domain is R, and domain and range have to be found.

- Domain, in this case, will be all the real values of x for which y is real
- Range is all the real values of y corresponding to values of x in the domain

Rules to find Domain

- If domain of f(x) is A and domain of g(x) is B, then domain of f(x)+g(x), f(x) - g(x), f(x) . g(x) is A ∩ B.
- For domain of f(x)/g(x), remove values of x for which g(x)=0, from A ∩ B.
- Domain of expressions of type $\sqrt{f(x)}$, we take the common values between A and values of x for which f(x) ≥ 0.
- Domain of polynomial function is R.
- Graphical method: we can also find domain if only the graph of function is given. We will learn this through the help of solved examples.

Methods to find Range

- Simple manipulations
- For range of y = f(x), we can first express x as a function of y: x = g(y). Now the domain of x = g(y) is same as the range of y = f(x)
- Graphical method: we can also find the range if only the graph of the function is given.

One-One (Injective) Function: A function f is one-one if different elements of A map to different elements of B.

$$f(a_1) = f(a_2) \implies a_1 = a_2 \quad \text{for all } a_1, a_2 \in A$$

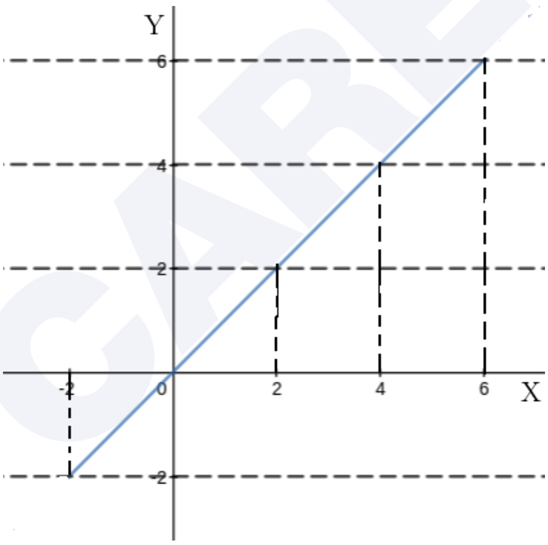
A function f : X ⟶ Y is called a one-one (or injective) function, if different elements of X have different images in B. i.e. no two elements of set X can have the same image.

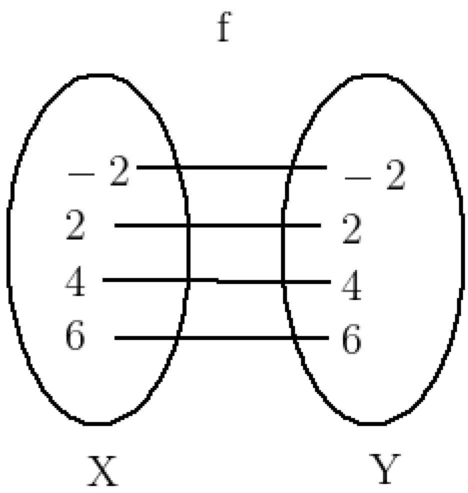
Consider,

f : X ⟶ Y, function given by y = f(x) = x , and

X = {-2, 2, 4, 6} and Y = {-2, 2, 4, 6},

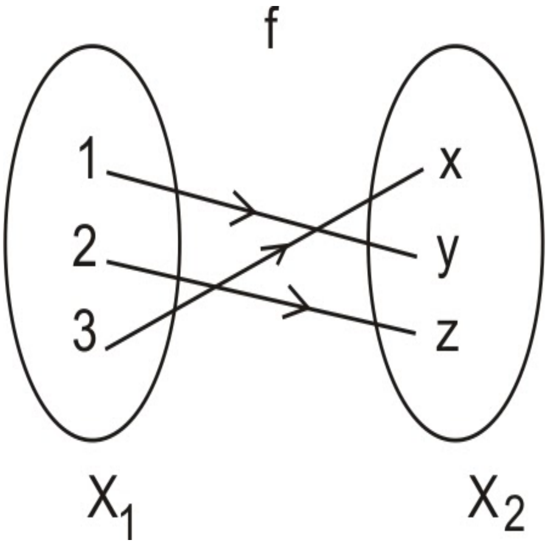
Graphically it can be shown that for every x, there is a unique y (or no y has more than one x corresponding to it) as below and hence it is one-one.





Now, consider, X1 = {1,2,3} and X2 = {x,y,z}

f : X1 ⟶ X2



Method to check One-One Function

1. If $x_1, x_2 \in X$, then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
2. A function is one - one iff no line parallel to X-axis meets the graph of function at more than one point.
3. Even degree polynomials are NOT one-one functions

Number of One - One Function

If A and B are finite sets having element m and n respectively, then the number of one-one function from A to B is

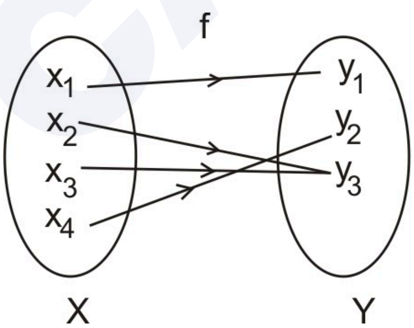
$$= \begin{cases} {}^nP_m & if \;\; n \geq m \\ 0 & if \;\; n < m \end{cases}$$

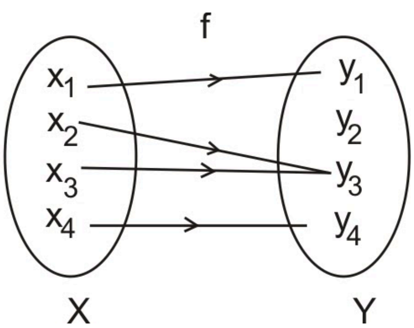
Many One function:

A function f : X $\rightarrow$ Y is called a many one function, if two or more elements of set X have the same image in set Y,

Or we can say that if f : X $\rightarrow$ Y is many- one if it is not one-one function.

To check it graphically a line parallel to x-axis cuts the curve at more than one point.





Both are many one, as in both there are two elements x_2 , x_3 which corresponds to the same image y_3, i.e. $f(x_2) = f(x_3) = y_3$

Method to check many-one

Check whether the function is one-one or not. If the function is not one-one then it is a many-one function.

Onto (Surjective) Function: A function f is onto if every element of B is an image of some element in A.

$\forall b \in B, \exists a \in A$ such that $f(a) = b$

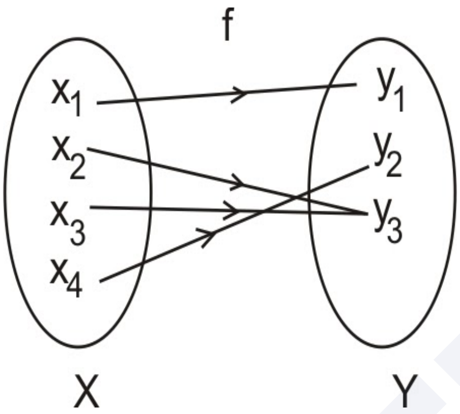
A function f : X → Y is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every y ∈ Y, there exists an element x in X such that f(x) = y

Hence, **Range = co-domain** for an onto function

Some examples of onto function

Consider, X = {x_1, x_2, x_3, x_4} and Y = {y_1, y_2, y_3}

f : X ⟶ Y



As every element in Y has a pre-image in X, so it is an onto function

Method to show onto or surjective

Find the range of y = f(x) and show that range of f(x) = co-domain of f(x)

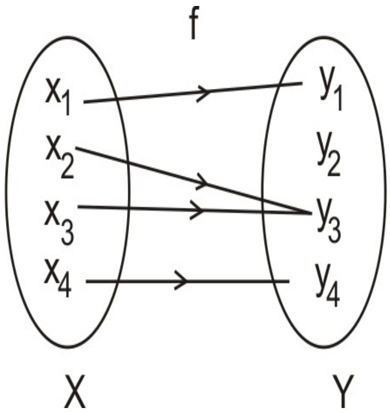
Into Function: A function f is into if some elements of B are not images of any element in A.

$f(A) \subsetneq B$

A function f : X ⟶ Y is said to be an into function if there exists an element in Y having no pre-image in A.

In other words, if f : X ⟶ Y is not onto mapping then it is an into mapping.

e.g.



As the element y_2 in codomain does not have a pre-image in domain, so it is into function

NOTE: If a function is not onto, then it is into and

If a function is not into, then it is onto.

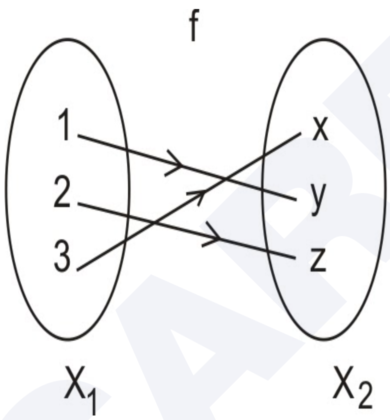
Bijective Function

A function f : X $\rightarrow$ Y is said to be bijective, if f is both one-one and onto (meaning it is both injective and surjective)

Consider, X_1 = {1,2,3} and X_2 = {x,y,z}

Eg

f : $X_1 \longrightarrow X_2$



The number of bijective function:

If f(x) is bijective, and the function is from a finite set A to a finite set B, then
$n\left(A\right)=n\left(B\right)=m\left(Say\right)$

And, the number of Bijective functions= m!

Equal Functions

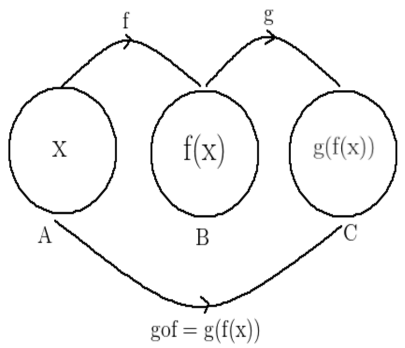
The two functions f and g are said to be equal if

- Domain(f) = Domain(g)
- Co-domain(f) = Co-domain(g), and
- f(x)=g(x) for all x belonging to domain

Composition of Functions: If $f : A \to B$ and $g : B \to C$, the composition $g \circ f : A \to C$ is defined as:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in A$$

Let f: A $\to$ B and g : B $\to$ C be two functions. Then the composition of f and g is denoted by gof and defined as the function gof : A $\to$ C given by gof (x) = g(f (x))



Properties of composition:

- In general fog ≠ gof (Not commutative)
- fo(goh) = (fog)oh (Associative)
- If f and g are bijections then fog and gof are also bijections
- The composition of any function with the identity function is the function itself. If $f : A \to B,$ then $foI_A = I_Bof = f$

Inverse of a function

Function f : X → Y is an invertible function if it is one-one and onto

Also its inverse g is defined in the following way

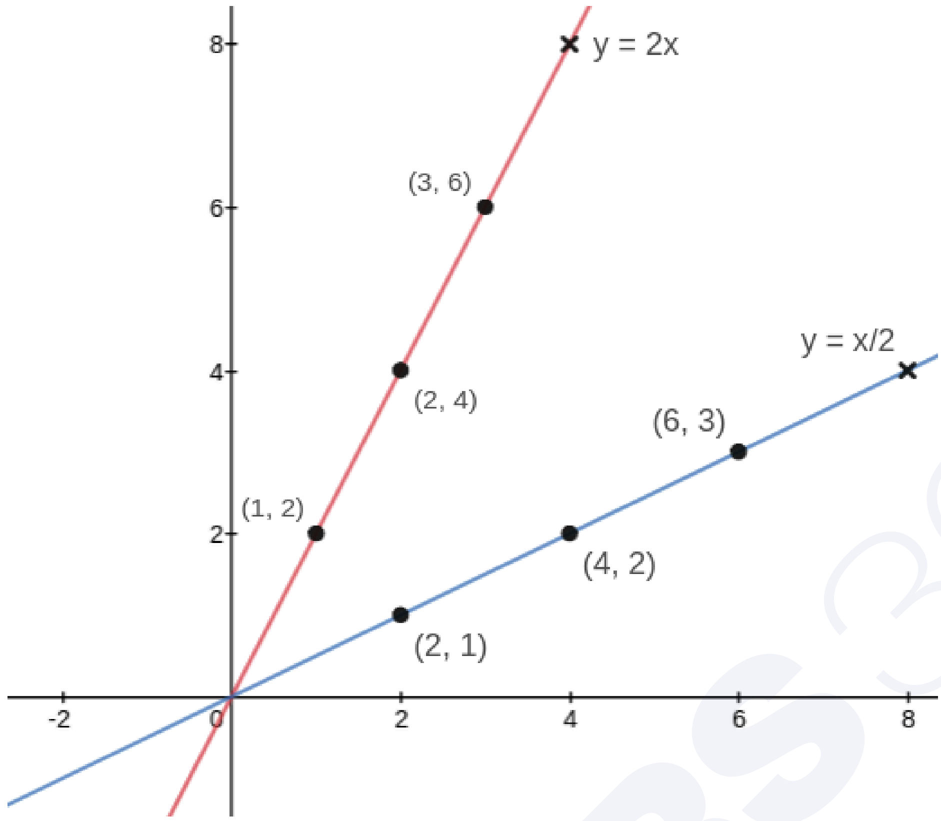
g : Y → X such that if f(a) = b, then g(b) = a

The function g is called the inverse of f and is denoted by f^{-1} .

Let us consider a one-one and onto function f with domain A and co-domain B. Where, A={1,2,3,4} and B={2,4,6,8} and f: A → B is given f(x)=2x, then write f and f^{-1} as a set of ordered pairs.

So, f ={(1,2) (2,4) (3,6) (4,8)}

And f^{-1} = {(2,1) (4,2) (6,3) (8,4)}



In above definition domain of f = {1,2,3,4} = range of f^{-1}

Range of f = {2,4,6,8} = domain of f^{-1}.

Steps to find the inverse of a function:

i) First we write f(x) as y and equate y=f(x), where f(x) is a function in x

ii) Then we separate the variable x as the dependent variable and express it in terms of y by assuming y as the independent variable

iii) Then we write g(y)=x where g(y) is a function in y

iv) And finally, we replace every y by x

Properties of an inverse function

i) The inverse of a bijection is unique.

ii) if f: A → B is a bijection and g: B → A is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.

iii) inverse of a bijection is also a bijection.

iv) If f: A → B and g: B → C are two bijections, then $(\mathrm{got})^{-1} = \mathrm{f}^{-1}\mathrm{og}^{-1}$

v) The graphs of f and its inverse function, are mirror images of each other in the line y = x.

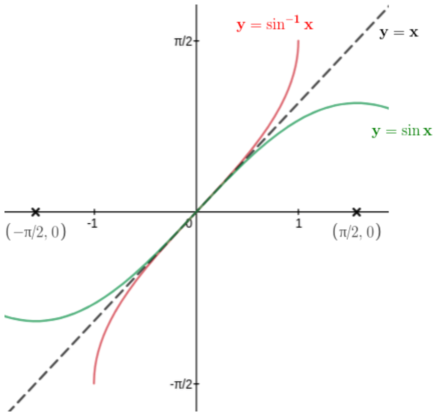
Inverse Trigonometric Functions

The trigonometric are many-one function in their actual domain. For inverse of trigonometric functions to be defined, the actual domain of trigonometric function must be restricted to make one-one function

y = sin^{-1}(x)

The function y = sin(x) is many one so it is not invertible. Now consider the small portion of the function

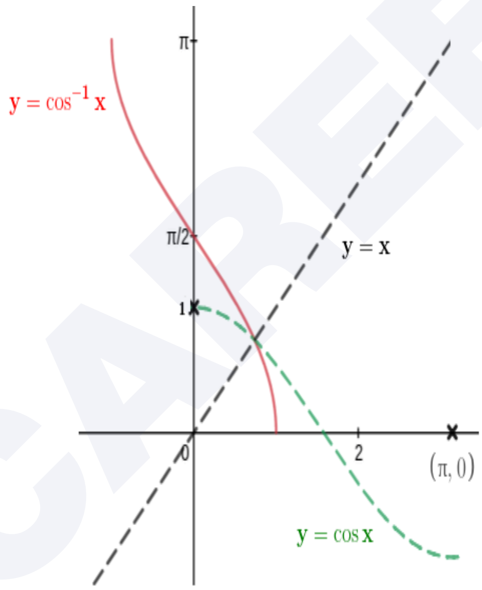
$$y = \sin x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{and} \quad y = [-1, 1]$$



Which is strictly increasing, Hence, this is one-one and its inverse is y = sin^{-1}(x)

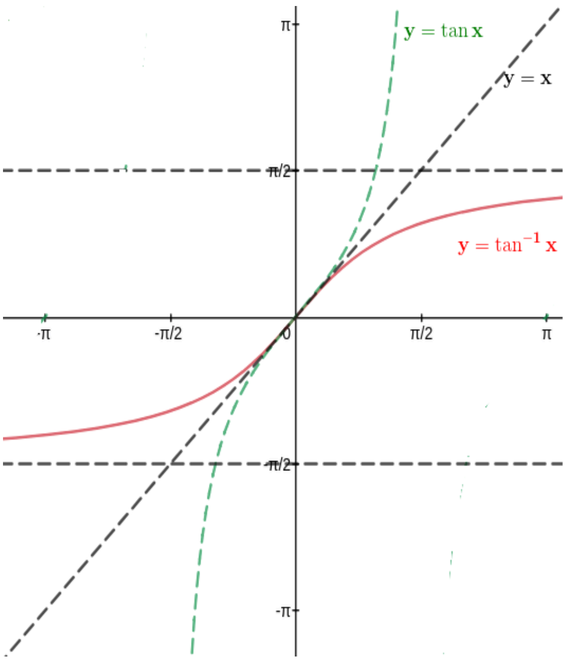
$$\text{Domain is } [-1, 1] \text{ and Range is } \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

y = cos^{-1}(x)



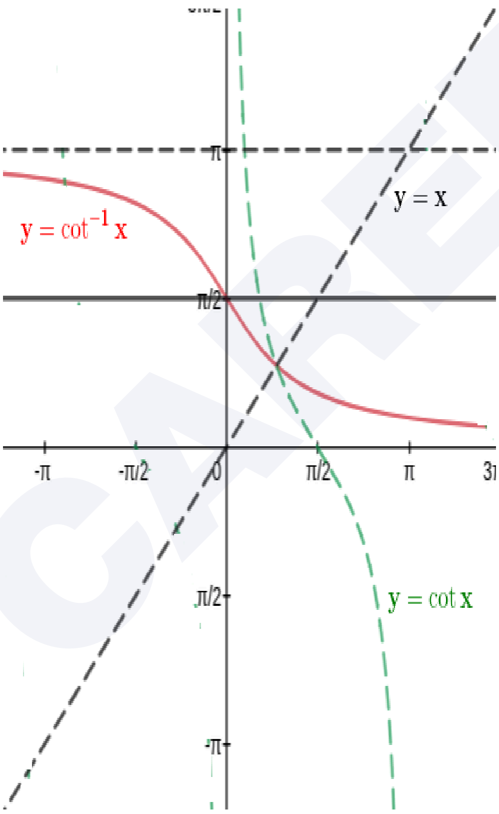
$$\text{Domain is } [-1, 1] \text{ and Range is } [0, \pi]$$

y = tan^{-1}(x)



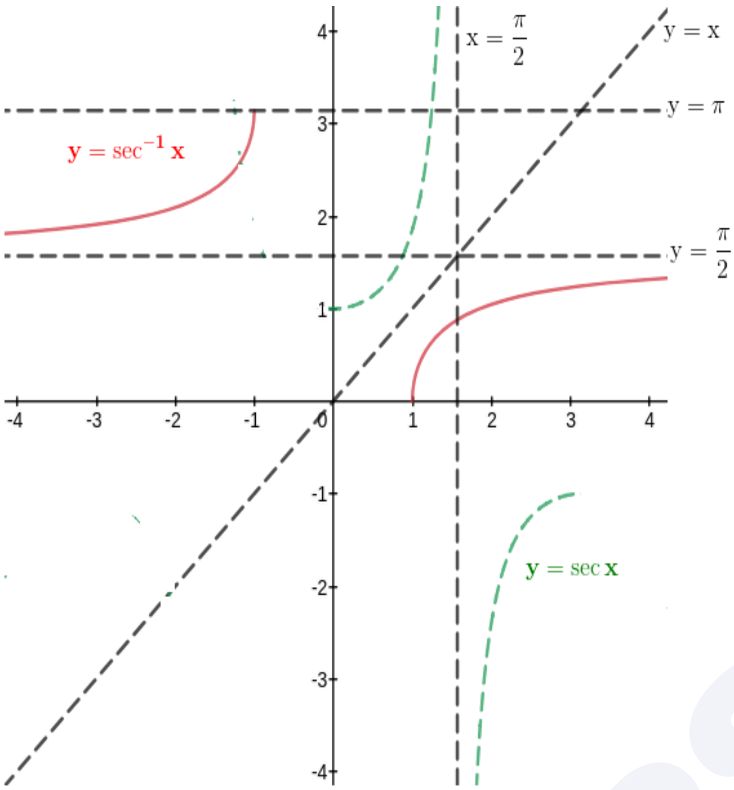
Domain is $\mathbb{R}$ and Range is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

y = cot^{-1}(x)



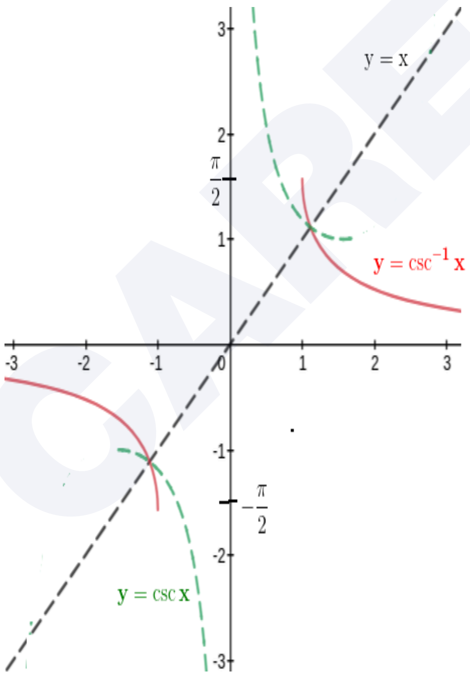
Domain is $\mathbb{R}$ and Range is $(0, \pi)$

y = sec^{-1}(x)



Domain is $\mathbb{R} - (-1, 1)$ and Range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

y = cosec^{-1}(x)

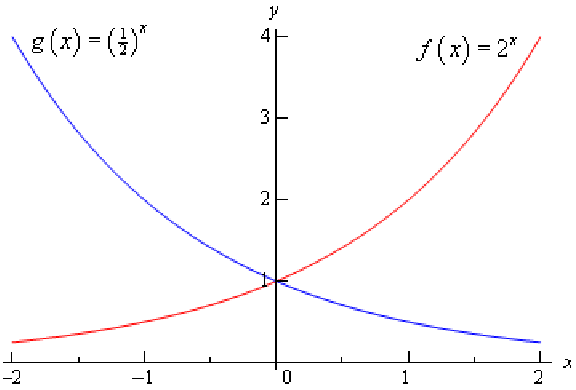


Domain is $\mathbb{R} - (-1, 1)$ and Range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Transcendental functions: the functions which are not algebraic are called transcendental functions. Exponential, logarithmic, trigonometric and inverse trigonometric functions are transcendental functions.

Exponential Function: function f(x) such that $\mathrm{f(x)} = \mathrm{a^x}$ is known as an exponential function.

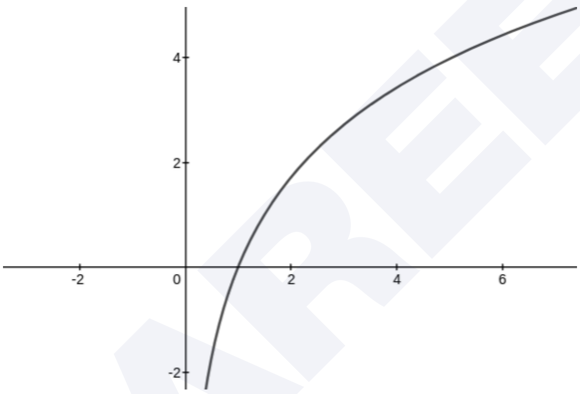
$\mathrm{base:}\quad \mathrm{a} > 0, \mathrm{a} \neq 1$
$\mathrm{domain: x} \in \mathbb{R}$
$\mathrm{range: f(x)} > 0$



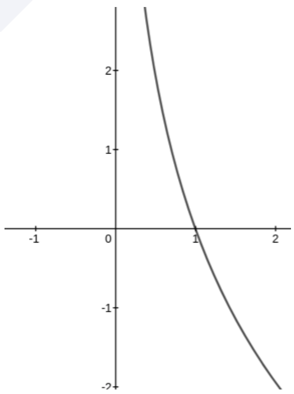
Property : If $a^x = a^y$,then x = y

Logarithmic function: function f(x) such that $f(x) = \log_a(x)$ is called logarithmic function

$\mathrm{base:}\quad \mathrm{a} > 0, \mathrm{a} \neq 1$
$\mathrm{domain: x} > 0$
$\mathrm{range: f(x)} \in \mathbb{R}$



If a > 1



If 0 < a < 1

Properties of Logarithmic Function

1. $\log_e(ab) = \log_e a + \log_e b$

2. $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b$

3. $\log_e a^m = m \log_e a$

4. $\log_\mathrm{a} \mathrm{a} = 1$

5. $\log_{b^m} a = \frac{1}{m} \log_b a$

6. $\log_b a = \frac{1}{\log_a b}$

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7. $\log_b a = \frac{\log_m a}{\log_m b}$

8. $a^{\log_a m} = m$

9. $a^{\log_c b} = b^{\log_c a}$

10. $\log_m a = b \Rightarrow a = m^b$

Vertical shift f(x) → f(x) ± a

A vertical shift of a function occurs if we add or subtract the constant to the function y = f(x)

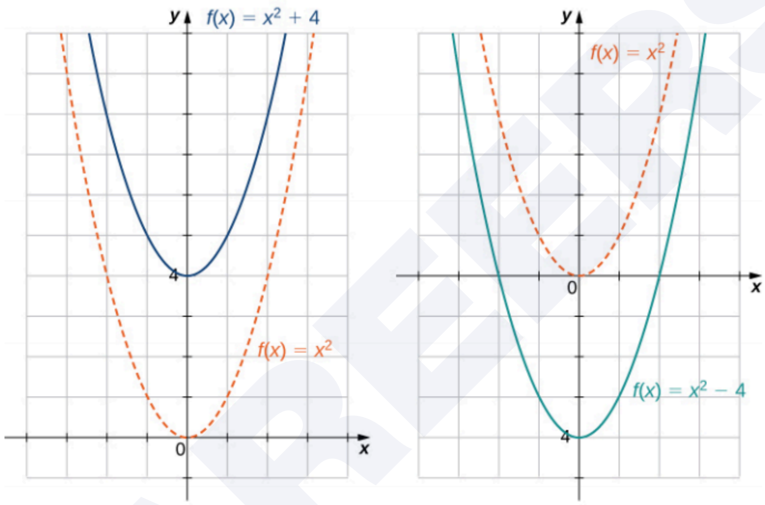
For a > 0, the graph of y = f(x) + a is obtained by shifting the graph of f(x) upwards by 'a' units,

whereas the graph of y = f(x) − a is obtained by shifting the graph of f(x) downwards by 'a' units.

For Example:

The graph of the function $f(x)=x^2 + 4$ is the graph of $f(x)=x^2$ shifted up by 4 units;

The graph of the function $f(x)=x^2 - 4$ is the graph of $f(x)=x^2$ shifted down by 4 units.



Horizontal shift: f(x) → f(x±a)

A horizontal shift of a function occurs if we add or subtract the same constant to each input x.

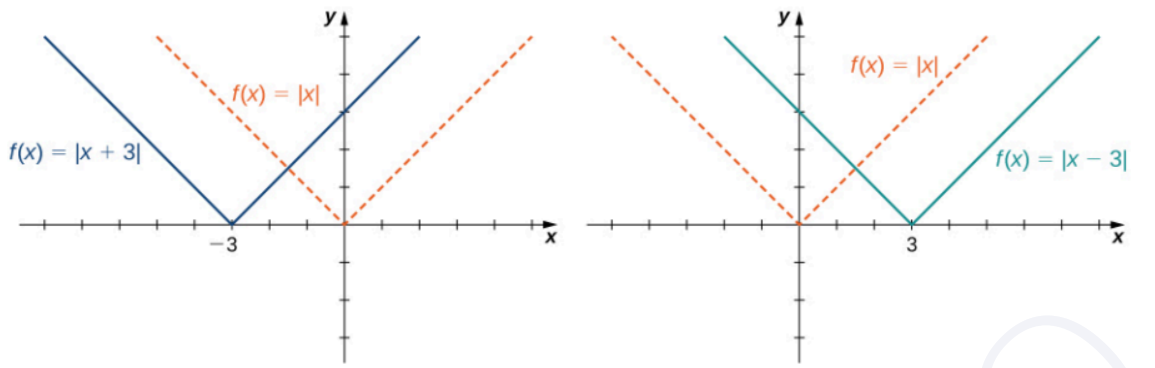
For a > 0, the graph of y = f(x + a) is obtained by shifting the graph of f(x) to the left by 'a' units.

The graph of y = f(x − a) is obtained by shifting the graph of f(x) to the right by 'a' units.

For Example

f(x) = |x + 3|

The graph of f(x) = |x + 3| is the graph of y = |x| shifted leftwards by 3 units. Similarly, the graph of f(x) = |x − 3| is the graph of y = |x| shifted rightward by 3 units



Chapter 2: Matrices

Order of a Matrix: A matrix with m rows and n columns has order $m \times n$.

Triangular Matrix:

Upper Triangular: All elements below the main diagonal are zero.

Lower Triangular: All elements above the main diagonal are zero.

Addition and Subtraction of Matrices: Possible only if matrices have the same order.

$$A = [a_{ij}], B = [b_{ij}] \Rightarrow A \pm B = [a_{ij} \pm b_{ij}]$$

Properties of Matrix Addition:

Commutative: $A + B = B + A$

Associative: $(A + B) + C = A + (B + C)$

Additive Identity: $A + O = A$

Additive Inverse: $A + (-A) = O$

Scalar Multiplication:

$$kA = [ka_{ij}]$$

Matrix Multiplication: $A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Properties of Matrix Multiplication:

Associative: $(AB)C = A(BC)$

Distributive: $A(B + C) = AB + AC$

Not commutative: $AB \neq BA$ in general

Transpose of a Matrix: $(A^T)_{ij} = A_{ji}$

Symmetric and Skew-Symmetric Matrices:

Symmetric: $A^T = A$

Skew-Symmetric: $A^T = -A$

Properties:

$A + A^T$ is symmetric

$A - A^T$ is skew-symmetric

Trace of a Matrix:

$\mathrm{Tr}(A) = \sum a_{ii}$ (sum of diagonal elements)

Chapter 3: Determinants

Singular Matrix: $|A| = 0$

Non-Singular: $|A| \neq 0$

Minor and Cofactor:

Minor M_{ij}: Determinant of matrix obtained by deleting i-th row and j-th column

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

Properties of Determinants:

$|A^T| = |A|$

$|AB| = |A||B|$

If two rows/columns are equal, $|A| = 0$

Interchanging two rows/columns changes sign

Adjoint of a Matrix:

$\mathrm{adj}(A) = [C_{ij}]^T$ (transpose of cofactor matrix)

Properties of Adjoint:

$A \cdot \mathrm{adj}(A) = |A| I$

$\mathrm{adj}(AB) = \mathrm{adj}(B) \cdot \mathrm{adj}(A)$

Inverse of a Matrix:

If $|A| \neq 0$, then

$A^{-1} = \frac{\text{adj}(A)}{|A|}$

Inverse of a 3×3 Matrix: Use $A^{-1} = \frac{\text{adj}(A)}{|A|}$ after finding cofactors and adjoint.

Properties of Inverse:

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

Multiplication of Determinants:

$|AB| = |A||B|$

System of Linear Equations:

Matrix Form: $AX = B$

If A is non-singular, unique solution: $X = A^{-1}B$

Cramer's Rule:

For $AX = B$, $x_i = \frac{\text{Det}(A_i)}{\text{Det}(A)}$ where A_i is matrix formed by replacing i-th column by B

System of Homogeneous Linear Equations:

$AX = 0$

Always has trivial solution $X = 0$

Non-trivial solution exists if $|A| = 0$

System of Equations using Matrix Method:

Solve $AX = B \Rightarrow X = A^{-1}B$ if A is invertible

Chapter 4: Continuity and Differentiability

Algebra of Continuous Functions:
If $f(x)$ and $g(x)$ are continuous at $x = a$, then:
$f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$ (if $g(a) \neq 0$) are also continuous at $x = a$.

Definition of Continuity:
$f(x)$ is continuous at $x = a$ if:
$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Differentiability Condition:
If $f(x)$ is differentiable at $x = a$, then it is also continuous at $x = a$ (converse not always true).

Derivatives of Functions

Basic Derivatives:

$\frac{d}{dx}(x^n) = nx^{n-1}$
$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(a^x) = a^x \ln a$

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

Chain Rule (Composite Functions):

If $y = f(g(x))$, then:
$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Logarithmic Differentiation:

If $y = f(x)^{g(x)}$, then take $\ln$ on both sides:
$\ln y = g(x) \ln f(x)$
Then differentiate:
$\frac{dy}{dx} = f(x)^{g(x)} \left(\frac{g(x) f'(x)}{f(x)} + g'(x) \ln f(x) \right)$

Implicit Differentiation:

If x and y are related implicitly (e.g., $x^2 + y^2 = 1$), differentiate both sides with respect to x using chain rule for y:
$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \Rightarrow 2x + 2y\frac{dy}{dx} = 0$

Higher Order Derivatives:

Second derivative:
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Rate of Change:
If $y = f(x)$, then rate of change of y with respect to x is:
$\frac{dy}{dx}$

Monotonicity:

- $f'(x) > 0 \Rightarrow$ Function is increasing
- $f'(x) < 0 \Rightarrow$ Function is decreasing

Maxima and Minima Conditions:

- $f'(x) = 0$, and
- If $f''(x) < 0 \Rightarrow$ Maxima
- If $f''(x) > 0 \Rightarrow$ Minima

Rolle's Theorem (conditions):
If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then $\exists c \in (a, b)$ such that: $f'(c) = 0$

Lagrange's Mean Value Theorem:
If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then $\exists c \in (a, b)$ such that:
$f'(c) = \frac{f(b)-f(a)}{b-a}$

Chapter 5: Integral Calculus

Basic Standard Integrals:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$\int \frac{1}{x} dx = \ln|x| + C$

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \sin x dx = -\cos x + C$

$\int \cos x dx = \sin x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \csc^2 x dx = -\cot x + C$

$\int \sec x \tan x dx = \sec x + C$

$\int \csc x \cot x dx = -\csc x + C$

Integrals of Inverse Trigonometric Functions:

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

Integration by Parts:

$\int u \cdot v dx = u \int v dx - \int \left(\frac{du}{dx} \cdot \int v dx \right) dx$

Integration by Substitution:

$\int f(g(x)) \cdot g'(x) dx = \int f(t) dt$, where $t = g(x)$

Integration Using Trigonometric Identities:

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\int \sin^m x \cos^n x dx$ and $\int \tan^m x \sec^n x dx$ use reduction techniques

Reduction Formula:

$I_n = \int (\sin x)^n dx$ or $\int (\cos x)^n dx$

$I_n = \frac{n-1}{n} I_{n-2}$ (for even powers)

Integration Using Euler's Substitution:

Use when integrand involves $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, or $\sqrt{x^2 + a^2}$

Definite Integration:

$\int_a^b f(x) dx = F(b) - F(a)$

Properties of Definite Integrals:

$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

$\int_a^a f(x) dx = 0$

$\int_a^b f(x) dx = -\int_b^a f(x) dx$

Even and Odd Function Properties:

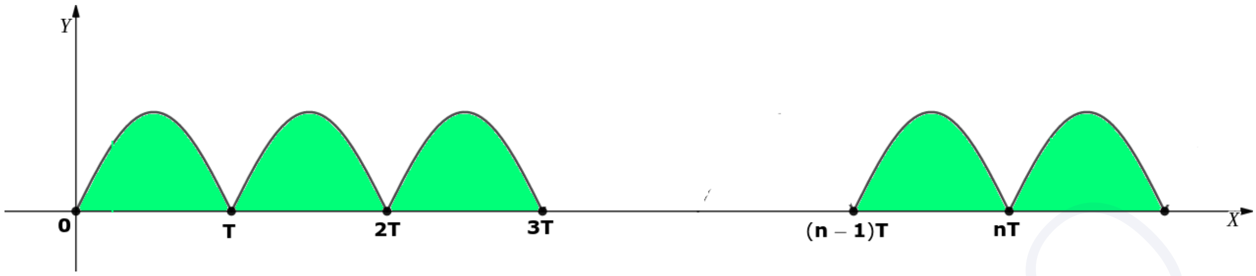
If $f(-x) = f(x)$ (even), then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

If $f(-x) = -f(x)$ (odd), then $\int_{-a}^a f(x) dx = 0$

Periodicity:

If f(x) is a periodic function with period T, then the area under f(x) for n periods would be n times the area under f(x) for one period, i.e.

$$\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$$



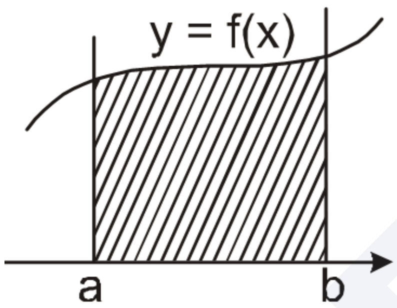
King's Property:

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Definite Integral as Limit of Sum:

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a+r\cdot h)\cdot h$, where $h = \frac{b-a}{n}$

Newton-Leibniz Theorem:



If $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$

Area Under Curve:

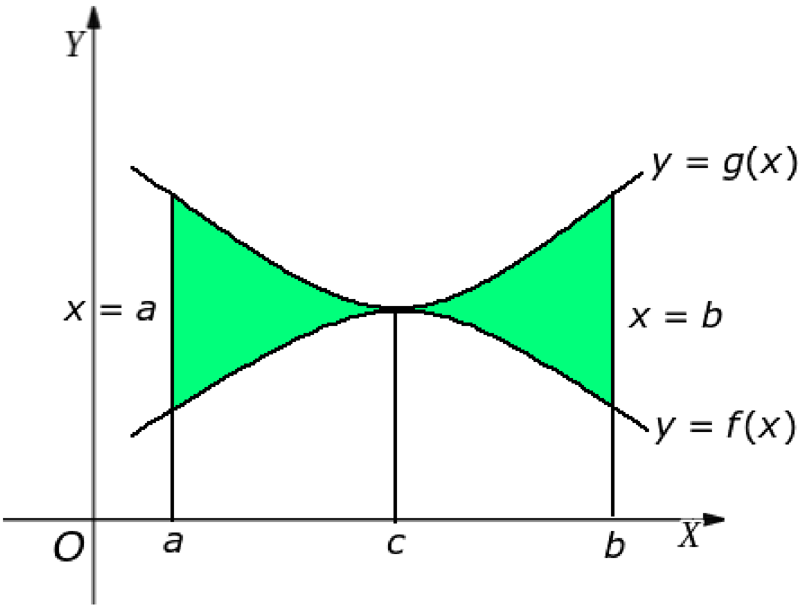
$\text{Area} = \int_a^b [f(x) - g(x)]dx$, where $f(x) \geq g(x)$

If bounded by curves $y^2 = f(x)$ or $x^2 = f(y)$, use proper variable change.

Area Bounded by Curves When Intersects at More Than One Point

Area bounded by the curves *y = f(x), y = g(x) which* intersect each other in the interval [a, b]

First find the point of intersection of these curves *y = f(x)* and *y = g(x)* by solving the equation f(x) = g(x), let the point of intersection be x = c



Area of the shaded region

$$= \int_a^c \{f(x) - g(x)\}dx + \int_c^b \{g(x) - f(x)\}dx$$

Chapter 6: Differential Equations

Order and Degree of Differential Equation:
The order is the highest order derivative present. The degree is the power of the highest derivative (after making it polynomial in derivatives).

Variable Separable Form:
If the equation can be written as $\frac{dy}{dx} = f(x)g(y)$
Then the solution is: $\int \frac{1}{g(y)}, dy = \int f(x), dx$

Homogeneous Differential Equation:
If $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
Then substitution: $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Linear Differential Equation:
Standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
Integrating Factor: $IF = e^{\int P(x),dx}$
Solution: $y \cdot IF = \int Q(x) \cdot IF, dx + C$

Bernoulli's Equation:
Form: $\frac{dy}{dx} + P(x)y = Q(x)y^n$
Divide by y^n and substitute: $z = y^{1-n}$, then reduce to linear form.

Exact Differential Equation:
Form: $M(x, y)dx + N(x, y)dy = 0$

Check: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution: $\int M, dx + \int \left(N - \frac{\partial}{\partial y}\left(\int M, dx\right)\right) dy = C$

Orthogonal Trajectories:

Given family of curves: $\frac{dy}{dx} = f(x, y)$

Then orthogonal trajectories satisfy: $\frac{dy}{dx} = -\frac{1}{f(x,y)}$

Chapter 7: 3D Geometry

Direction Cosines of a Vector $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

$$l = \frac{a}{|\vec{a}|}, m = \frac{b}{|\vec{a}|}, n = \frac{c}{|\vec{a}|}$$

Direction Ratios:

Proportional to direction cosines: $a : b : c$

Angle Between Two Lines:

$$\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$$

Perpendicular Distance from Point to Line:

$$D = \frac{|\vec{b} \times (\vec{r} - \vec{a})|}{|\vec{b}|}$$

Shortest Distance Between Two Skew Lines:

$$D = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$$

Equation of a Plane in Normal Form:

$$\vec{r} \cdot \hat{n} = p$$

where $\hat{n}$ is a unit normal vector and p is perpendicular from origin.

Plane Perpendicular to Vector $\vec{n}$ Through Point $\vec{a}$:

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

Plane Through Three Non-Collinear Points:

Use determinant:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Plane Through Point and Parallel to Vectors $\vec{b}_1, \vec{b}_2$:

$(\vec{r} - \vec{a}) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$

Angle Between Two Planes: $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Family of Planes Passing Through Line of Intersection:

Plane: $P_1 + \lambda P_2 = 0$, where $P_1 = 0$ and $P_2 = 0$ are two planes.

Distance of a Point (x_1, y_1, z_1) from Plane $ax + by + cz + d = 0$:

$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Image of a Point in a Plane: Use projection method:

$\vec{P'} = \vec{P} - 2\left(\frac{\vec{n} \cdot (\vec{P} - \vec{A})}{|\vec{n}|^2}\right)\vec{n}$

Plane Bisecting Angle Between Two Planes:

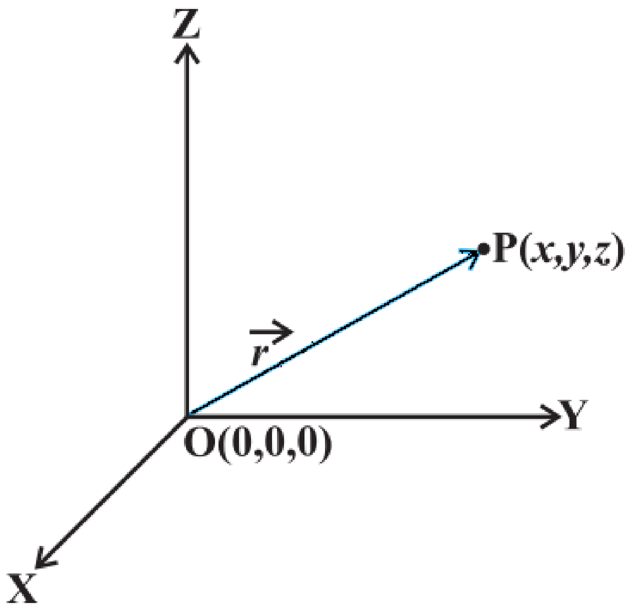
$\frac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{a'x + b'y + c'z + d'}{\sqrt{a'^2 + b'^2 + c'^2}}$

Coplanarity of Two Lines: Lines are coplanar if

$[(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)] = 0$

Chapter 8: Vector Algebra

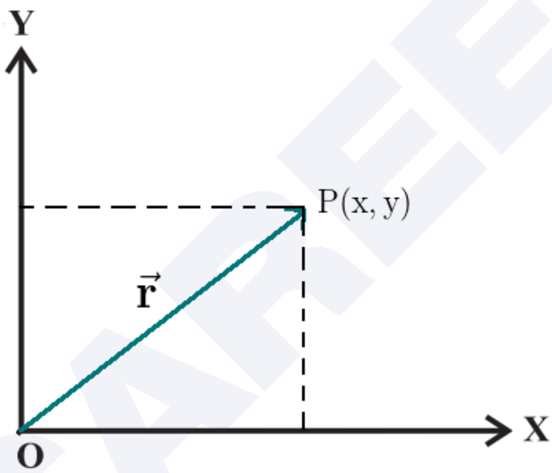
Distance of a Vector from Origin:



Where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors parallel to positive X-axis, Y-axis and Z-axis respectively.

If $\vec{r} = \langle x, y, z \rangle$, then distance from origin = $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Position Vector:



Position vector of point $P(x, y, z)$ from origin is $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Magnitude of a Position Vector:

$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Types of Vectors:

1. Zero Vector: $\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

2. Unit Vector: $|\vec{a}| = 1$

3. Like and Unlike Vectors

4. Collinear Vectors

5. Coplanar Vectors

6. Equal Vectors: Same magnitude and direction

Projection of Vector $\vec{a}$ on $\vec{b}$:

$\operatorname{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Collinear Vectors:

$\vec{a}$ and $\vec{b}$ are collinear if $\vec{a} = k\vec{b}$ for some scalar k

Equal Vectors:

$\vec{a} = \vec{b}$ if they have same magnitude and direction

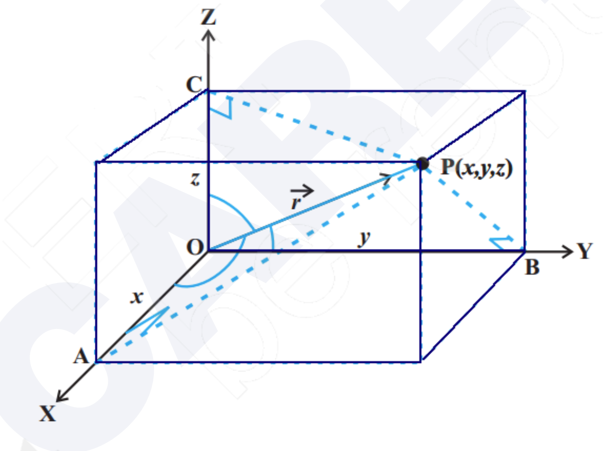
Coplanar Vectors:

Three vectors are coplanar if their scalar triple product is zero

$(\vec{a}\cdot(\vec{b}\times\vec{c})) = 0$

Direction Cosines (DCs)

Let r be the position vector of a point P(x, y, z). Then, direction cosines of vector r are the cosines of angles α, β and γ (i.e. cos α, cos β and cos γ) that the vector r makes with the positive direction of X, Y and Z -axes respectively. Direction cosines are usually denote by l, m and n respectively.



From the figure, note that, ΔOAP is a right angled triangle and thus, we have

$$
\cos \alpha=\frac{x}{r}(r \text { stands for }|r|)
$$

Similarly, from the right angled triangles OBP and OCP, We have,

$$
\cos \beta=\frac{y}{r} \text { and } \cos \gamma=\frac{z}{r}

`$$`

So we have the following results,

$$

\begin{aligned}

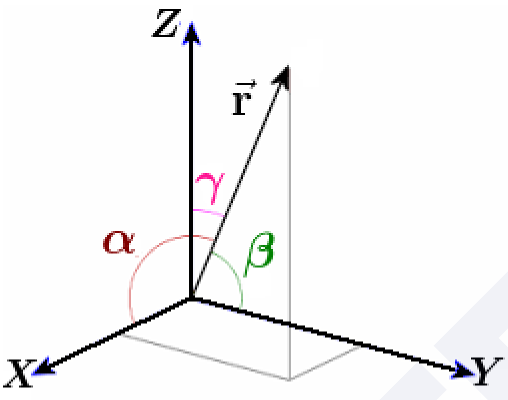
& \cos \alpha=l=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{|\mathbf{r}|}=\frac{x}{r} \

& \cos \beta=m=\frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{|\mathbf{r}|}=\frac{y}{r} \

& \cos \gamma=n=\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{|\mathbf{r}|}=\frac{z}{r}

\end{aligned}

$$



Also,

$$

\begin{aligned}

& l^2=\frac{x^2}{x^2+y^2+z^2} \

& m^2=\frac{y^2}{x^2+y^2+z^2} \

& n^2=\frac{z^2}{x^2+y^2+z^2}

\end{aligned}

$$

Add (i), (ii) and (iii)

$$

\begin{aligned}

& l^2+m^2+n^2=\frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2} \

& \Rightarrow l^2+m^2+n^2=1 \

& \Rightarrow \cos ^2 \alpha+\cos ^2 \beta+\cos ^2 \gamma=1

\end{aligned}

$$

The coordinates of the point P may also be expressed as (lr, mr,nr).

Direction ratios (DRs)

Direction Ratios are any set of three numbers that are proportional to the Direction cosines.

If l, m, n are DCs of a vector then $\lambda l, \lambda m, \lambda n$ are DRs of this vector, where a can take any real value.

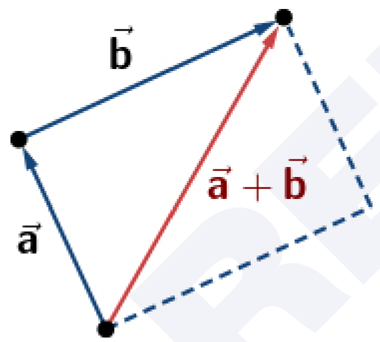
DRs are also denoted as a, b and c, respectively.

A vector has only one set of DCs, but infinite sets of DRs.

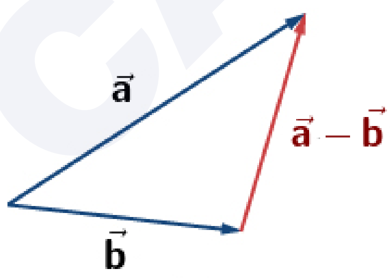
Note:
The coordinates of a point equal lr, mr and nr, which are proportional to the direction cosines. Hence the coordinates of a point are also its DRs.

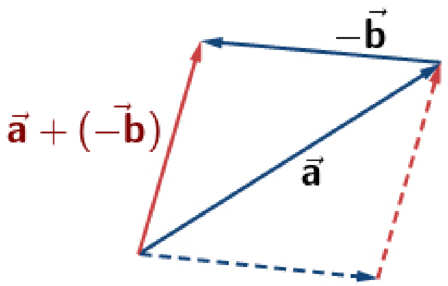
If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b and c are one of the direction ratios of the given vector. Also, if $a^2 + b^2 + c^2 = 1$, then a, b and c will be direction cosines of given vector.

Addition and Subtraction of Vectors:



(a)





$\vec{a} \pm \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \pm (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

Component Form of Vector from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$:

$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

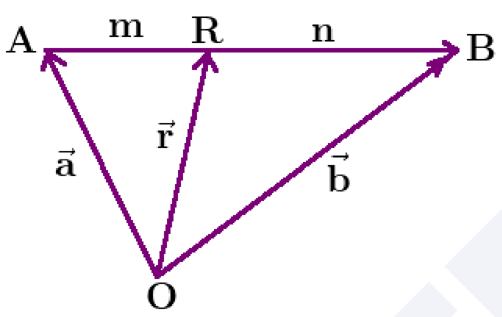
Multiplication of Vector by Scalar k:

$k\vec{a} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

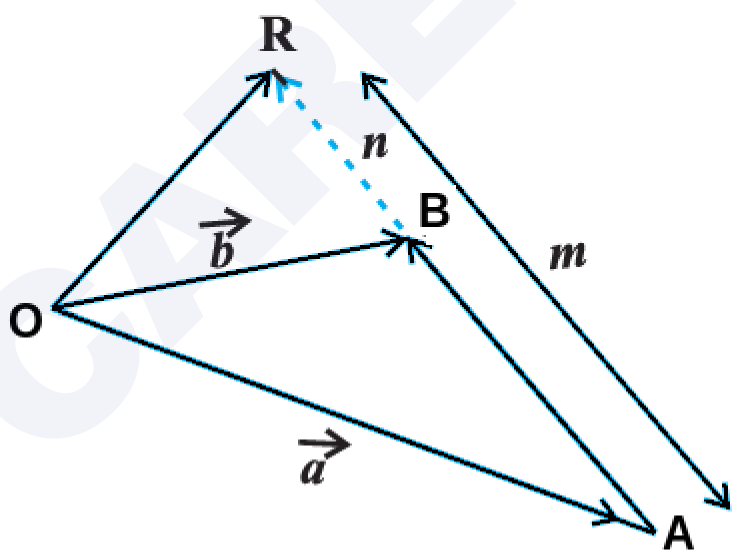
Section Formula (Vectors):

Internal Division

If R divides AB internally in the ratio m : n, then position vector of R is given by $\overrightarrow{\mathrm{OR}} = \frac{m\vec{b}+n\vec{a}}{m+n}$



External Division



If R divides AB externally in the ratio m : n, then position vector of R is given by $\overrightarrow{\mathrm{OR}} = \frac{m\vec{b}-n\vec{a}}{m-n}$

NOTE:

If R is the midpoint of AB, then m = n. And therefore, the midpoint R of $\overrightarrow{AB}$, will have its position vector as$\overrightarrow{\mathrm{OR}} = \frac{\vec{a}+\vec{b}}{2}$

Linear Combination of Vectors:

$\vec{r} = a\vec{u} + b\vec{v} + c\vec{w}$, where a, b, c are scalars

Dot Product (Scalar Product):

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Dot Product in Components:

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

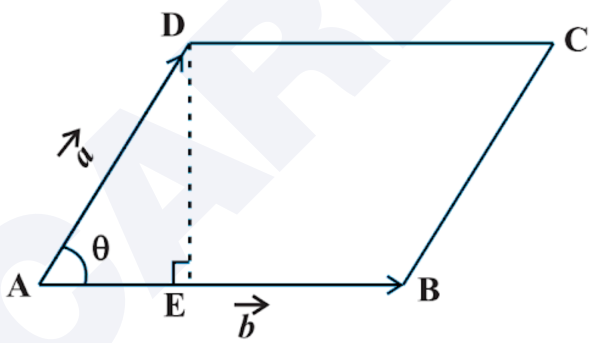
Cross Product (Vector Product):

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$

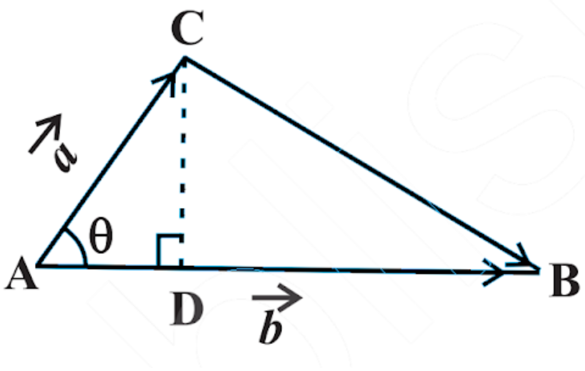
Cross Product in Components:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Geometric Interpretation of Vectors:



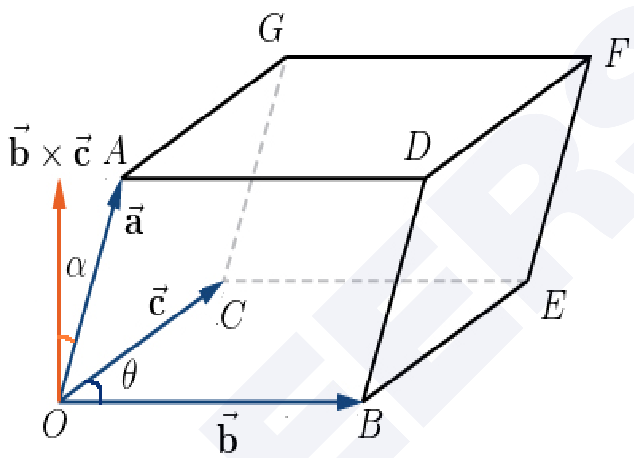
1. $|\vec{a} \times \vec{b}|$ = Area of parallelogram



2. $\frac{1}{2}|\vec{a} \times \vec{b}|$ = Area of triangle

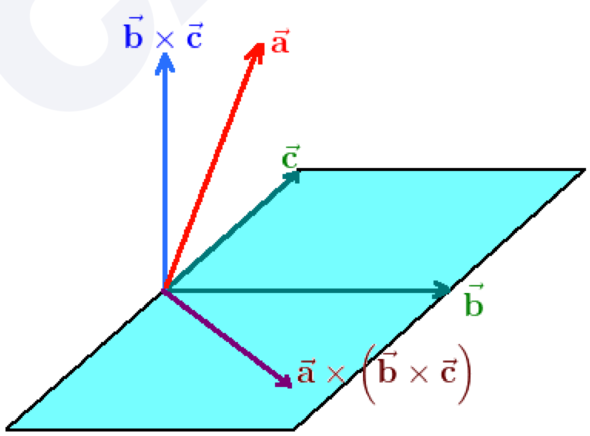
3. $|\vec{a} \cdot \vec{b}|$ = Projection of one vector on another times its magnitude

Scalar Triple Product (STP):



$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector Triple Product:



$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Lagrange's Identity:

$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

Chapter 9: Probability

Probability of an Event:

$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$

Range of Probability: $0 \leq P(E) \leq 1$

Complementary Event: $P(E') = 1 - P(E)$

Sure and Impossible Events: $P(S) = 1, P(\phi) = 0$

Addition Theorem (General): $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Addition Theorem (Mutually Exclusive Events): If $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$

Intersection of Two Events: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Conditional Probability:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) \neq 0$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

Multiplication Theorem of Probability:

$P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

Independent Events: Events A and B are independent if

$P(A \cap B) = P(A) \cdot P(B)$

Total Probability Theorem: If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, then for any event A:

$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)$

Bayes' Theorem: If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, and A is any event, then:

$P(B_k|A) = \frac{P(B_k) \cdot P(A|B_k)}{\sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)}$

Probability Distribution of a Random Variable: If X is a discrete random variable, its distribution is given by:

$P(X = x_i) = p_i$, where $0 \leq p_i \leq 1$ and $\sum p_i = 1$

Expected Value: $E(X) = \sum x_i \cdot p_i$

Variance: $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

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